Problem Corner

A Divisibility Chain Problem

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Notation. Let u and v be positive integers. By $u \mid v$ we mean "u is a divisor of v" and by $u \nmid v$ we mean: "u is not a divisor of v". For example, $4 \mid 12$, but $5 \nmid 12$.

Problem

Let a and b be positive integers such that

$$a \mid b^2, \quad b^2 \mid a^3, \quad a^3 \mid b^4, \quad b^4 \mid a^5, \quad a^5 \mid b^6, \quad \dots . \qquad (1)$$

Prove that $a = b$.

Solution

We make use of the following auxiliary result (such a preliminary step is also called a 'lemma'):

Lemma. *Let m and n be positive integers such that*

$$m \le 2n \le 3m \le 4n \le 5m \le 6n \le \cdots . \qquad (2)$$

Then $m = n$.

Proof of lemma. The inequalities $m \le 2n \le 3m \le 4n \le \cdots$ imply that $(2k-1)m \le 2kn$ for every positive integer k. Hence we have

$$\frac{m}{n} \le 1 + \frac{1}{2k-1} \quad \text{for every positive integer } k.$$

Since

$$\frac{1}{2k-1} \to 0 \quad \text{as } k \to \infty,$$

it follows that

$$\frac{m}{n} \le 1,$$

and so $m \le n$.

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The same inequalities also imply that $2kn \leq (2k+1)m$ for every positive integer k. Hence we have

$$\frac{n}{m} \leq 1 + \frac{1}{2k} \quad \text{for every positive integer } k.$$

Reasoning the same way as we did earlier, we conclude that

$$\frac{n}{m} \leq 1,$$

and so $n \leq m$.

Since $m \leq n$ and $n \leq m$, it follows that $m = n$. □

Solution of problem. The divisibility conditions imply that for any prime number p, if $p \mid a$ then $p \mid b$ as well; and in the same way, if $p \mid b$ then $p \mid a$ as well. Hence a and b are divisible by exactly the same set of primes.

Let p be any prime number dividing a, b. Let p^u be the highest power of p that divides a, and let p^v be the highest power of p that divides b. That is, we have $p^u \mid a$ but $p^{u+1} \nmid a$; and $p^v \mid b$ but $p^{v+1} \nmid b$. Here $u > 0$ and $v > 0$. Then from the given conditions we argue as follows:

- $a \mid b^2$, so $p^u \mid p^{2v}$, so $u \leq 2v$;
- $b^2 \mid a^3$, so $p^{2v} \mid p^{3u}$, so $2v \leq 3u$;
- $a^3 \mid b^4$, so $p^{3u} \mid p^{4v}$, so $3u \leq 4v$;
- $b^4 \mid a^5$, so $p^{4v} \mid p^{5u}$, so $4v \leq 5u$;

and so on. Hence:

$$u \leq 2v \leq 3u \leq 4v \leq 5u \leq \cdots.$$

Invoking the lemma proved above, we deduce that $u = v$. So the highest power of p that divides a is identical to the highest power of p that divides b.

Since the same is true for every prime number that divides a and b, it follows that a and b have identical prime factorization. This implies that $a = b$. □