Divergence and Convergence of Two Closely Related Infinite Series

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There are two kinds of infinite series: those that are *divergent*, and those that are *convergent*. The sum of the terms of a divergent series increases without limit as more terms are taken; the sum of the terms of a convergent series approaches a definite number as more terms are taken. A well-known divergent series is the *harmonic series*, which is the sum of the reciprocals of the positive integers, $\sum \frac{1}{n}$. It was first proved to be divergent by Nicole Oresme (1323-1382). Others gave proofs of the same statement, e.g., Cauchy, Augustus De Morgan, Johann Bernoulli, Euler, etc. Here I put forward my way for proving the divergence of $\sum_{n=1}^{\infty} 1/n$.

Theorem: $\sum_{n=1}^{\infty} 1/n$ is divergent.

Proof: Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, it follows that if $x > 0$, then $e^x \geq 1 + x$ or $x \geq \ln(1 + x)$. Substituting $x = 1/n$ where n is a positive integer, we get

$$\frac{1}{n} \geq \ln\left(1 + \frac{1}{n}\right) = \ln(n+1) - \ln n.$$

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Taking summation on both sides for $n = 1, 2, \ldots, k$, we get

$$\sum_{n=1}^{k} \frac{1}{n} > (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \cdots + (\ln (k+1) - \ln k)$$

$$\Rightarrow \sum_{n=1}^{k} \frac{1}{n} > \ln (1 + k) .$$

Taking limits on both sides, $k \longrightarrow \infty$, we get

$$\sum_{n=1}^{\infty} \frac{1}{n} > \lim_{k \to \infty} \ln (1 + k) = \infty.$$

Hence $\sum_{n=1}^{\infty} 1/n$ is divergent.

In the series considered above, all the terms are positive. Another kind of series is one where the terms alternate in sign; they are called *alternating series*. We now examine a well-known alternating series closely related to the harmonic series: $\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots.$

Theorem: $\sum_{n=1}^{\infty} (-1)^{n-1} / n$ is convergent.

Proof: Consider the following sequence of inequalities which are all clearly true:

$$\frac{1}{3} - \frac{1}{4} < \frac{1}{2} - \frac{1}{3}$$
$$\frac{1}{5} - \frac{1}{6} < \frac{1}{3} - \frac{1}{4}$$
$$\frac{1}{7} - \frac{1}{8} < \frac{1}{4} - \frac{1}{5}$$
$$\ldots\ldots\ldots$$

We also have, trivially:

$$1 - \frac{1}{2} = 1 - \frac{1}{2}.$$

Adding the corresponding sides of all these statements, we get

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \ldots$$
$$< \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots,$$

which yields, after simplifying the expression on the right side,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots < 1.$$

It follows that the sum of this alternating series is bounded above by 1. By converting the series into a sum of positive terms as follows,

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\ldots=\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\frac{1}{56}+\ldots,$$

we conclude that the series is convergent. (Mathematically, it is more accurate to say that the series is "conditionally convergent.")

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