Student Corner

A Property of a Class of Polygons

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Consider a polygon $\mathcal{P}$ with the following three features.

(1) It is simple, i.e., it does not intersect itself, and it has no holes. $\mathcal{P}$ is permitted to be non-convex, i.e., with indentations.

(2) The number of sides of $\mathcal{P}$ is of the form $4k + 2$.

(3) If the polygon is $P_1P_2P_3 \ldots P_{4k+2}$, then the $2k + 1$ 'main' diagonals P_1P_{2k+2}, P_2P_{2k+3}, …, $P_{2k+1}P_{4k+2}$ meet in a point.

Three examples of such polygons are shown in Figure 1. Note that a triangle with 3 concurrent cevians may be considered as a special case of such a polygon (see the figure in the middle; it may be regarded as a hexagon). Likewise for a quadrilateral with a line segment drawn through the point of intersection of the two diagonals (the third figure). All the polygons shown below have $4k + 2 = 6$ sides, i.e., with $k = 1$.

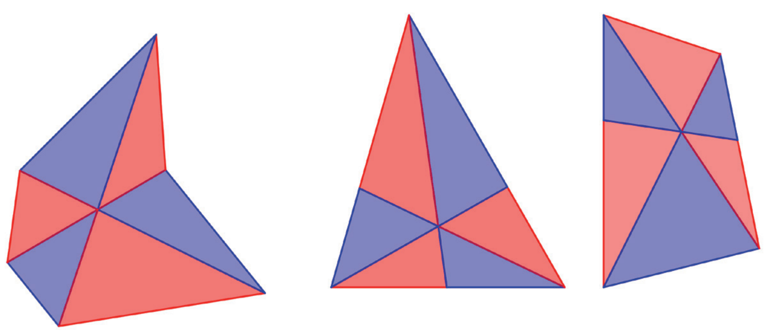

Figure 1

For such a polygon $\mathcal{P}$, let the $2k + 1$ main diagonals be drawn. These divide the interior of $\mathcal{P}$ into $4k + 2$ non-overlapping triangles. Let these triangles be coloured alternately blue and red, as shown.

Keywords: Simple polygon, diagonal, area

Theorem. *The product of the areas of the regions coloured blue is equal to the product of the areas of the regions coloured red.*

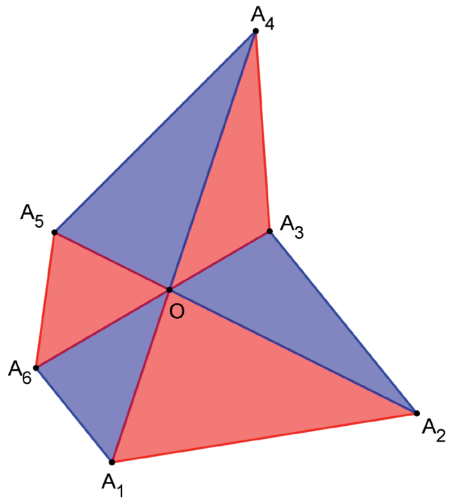

Figure 2

The case $k = 1$ is depicted in Figure 2. With reference to this figure, the claim is that

$$[OA_1A_2] \cdot [OA_3A_4] \cdot [OA_5A_6] = [OA_2A_3] \cdot [OA_4A_5] \cdot [OA_6A_1], \tag{1}$$

where the square brackets denote area.

Proof. Let the lengths of segments $\measuredangle OA_1$, $\measuredangle OA_2$, $\measuredangle OA_3$, …, $\measuredangle OA_6$ be a_1, a_2, a_3, …, a_6, respectively. Let the measures of $\measuredangle A_1OA_2$, $\measuredangle A_2OA_3$, $\measuredangle A_3OA_4$, …, $\measuredangle A_6OA_1$ be θ_1, θ_2, θ_3, …, θ_6, respectively. Then

$$[OA_1A_2] = \frac{1}{2} \cdot a_1 \cdot a_2 \cdot \sin\theta_1, \tag{2}$$

with similar expressions for the areas of the remaining triangles. It follows that

$$[OA_1A_2] \cdot [OA_3A_4] \cdot [OA_5A_6] = \frac{1}{2^3} \cdot a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdot a_6 \cdot \sin\theta_1 \cdot \sin\theta_3 \cdot \sin\theta_5, \tag{3}$$

$$[OA_2A_3] \cdot [OA_4A_5] \cdot [OA_6A_1] = \frac{1}{2^3} \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdot a_6 \cdot a_1 \cdot \sin\theta_2 \cdot \sin\theta_4 \cdot \sin\theta_6, \tag{4}$$

and equality (1) follows immediately because of the angle equalities $\theta_1 = \theta_4$, $\theta_2 = \theta_5$, $\theta_3 = \theta_6$.

The same reasoning works for any value of k. □

Now we see why n must be of the form $4k + 2$. The number of sides must clearly be even. Moreover, $n/2$ must be odd, else the regions corresponding to vertically opposite angles will receive the same colour and the stated result would not hold. Hence we write $n/2 = 2k + 1$, i.e., $n = 4k + 2$.

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