Dwelling on the Incircle

PRITHWIJIT DE

The purpose of this note is to encourage students and teachers alike to explore simple configurations in geometry and come up with interesting questions which can be answered by employing elementary knowledge of plane geometry.

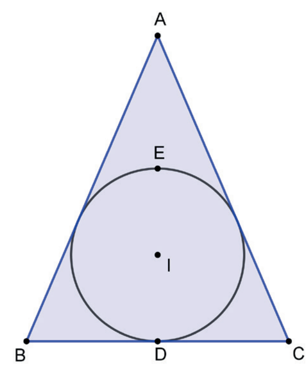

Figure 1

Given a triangle ABC with $AB = AC$ (see Figure 1). Let I be its incentre, and r its inradius. For such a triangle the circumcentre (O), centroid (G), orthocentre (H) and nine-point centre (N) lie on the line AI. It may so happen that any of these points lies on the incircle. Can we determine all such isosceles triangles ABC with $AB = AC$, up to similarity, for which any one of O, G, H and N lies on the incircle? Let us explore.

Keywords: Incircle, incentre, circumcentre, centroid, orthocentre, antipode

Let $AB = AC = x$ and $BC = y$. We consider each possibility in turn.

If the circumcentre lies on the incircle …. Suppose O lies on the incircle. Let D be the point of intersection of AI and BC and let E be the antipode of D on the incircle (i.e., the point diametrically opposite to D). Thus O can coincide with either D or E.

If O coincides with D, then $\measuredangle BAC = \frac{1}{2}\measuredangle BOC = 90°$, so ABC is a right-angled isosceles triangle.

If O coincides with E, then $OD = R\cos A = 2r$, so

$$AD = AO + OD = R + 2r,$$

where R is the radius of the circumcircle, and

$$AD = AI + ID = r\csc(A/2) + r.$$

Combining these we obtain

$$\frac{r}{R} = \frac{\sin(A/2)}{1-\sin(A/2)} = \frac{\cos A}{2}.$$

Using $\cos A = 1 - 2\sin^2(A/2)$ we reduce the last equation to

$$1 - 3\sin(A/2) - 2\sin^2(A/2) + 2\sin^3(A/2) = (1+\sin(A/2))(1 - 4\sin(A/2) + 2\sin^2(A/2)) = 0.$$

Since $0 < \sin(A/2) < 1$, the only admissible root of this equation is $\sin(A/2) = 1 - 1/\sqrt{2}$ whence $\measuredangle BAC = 34°$, approximately. See Figure 2.

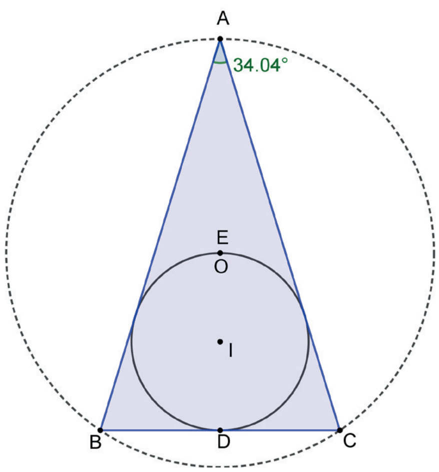

Figure 2

If the centroid lies on the incircle …. If G lies on the incircle, it must coincide with E. Hence $GD = 2r$. Since AD is the median we must have $GD = AD/3$. Therefore

$$2r = \frac{1}{3}\sqrt{x^2 - y^2/4}.$$

But

$$r = \frac{[ABC]}{s} = \frac{\frac{y}{2}\sqrt{x^2 - y^2/4}}{(2x+y)/2}.$$

Eliminating r from these relations, we get

$$6y = 2x + y, \quad \therefore \ \frac{y}{x} = \frac{2}{5},$$

and $\sin(A/2) = y/2x = 1/5$ and $\measuredangle BAC = 23°$, approximately. See Figure 3.

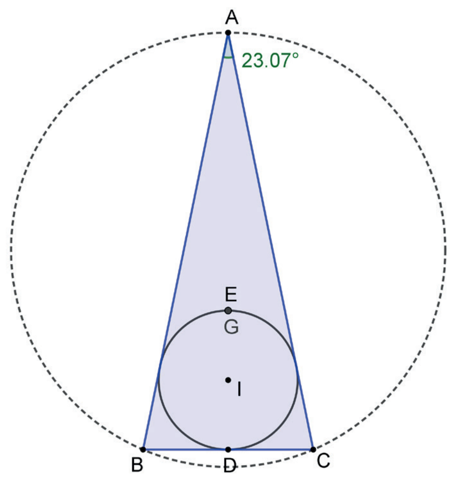

Figure 3

If the orthocentre lies on the incircle .... Similarly, if H lies on the incircle, then it has to coincide with E, so $HD = 2r$. Angle chasing gives $\measuredangle DAB = \measuredangle DBH$, therefore the triangles DAB and DBH are similar. Hence

$$\frac{BD}{AD} = \frac{HD}{BD}.$$

But $BD = y/2$, $AD = \sqrt{x^2 - y^2/4}$ and $HD = 2r = \dfrac{2y\sqrt{x^2 - y^2/4}}{2x+y}$. Therefore we obtain

$$y^2 = \frac{2y(4x^2 - y^2)}{2x+y}$$

whence $4x = 3y$ implying that $\sin(A/2) = y/2x = 2/3$ and $A = 84°$, approximately. See Figure 4.

If the nine-point centre lies on the incircle .... Lastly, we consider the case where N lies on the incircle. Recall that N is the centre of the circumcircle of the triangle (the *medial triangle*) whose vertices are the midpoints of the sides of ABC. Since the medial triangle of any triangle is similar to the original triangle with similarity ratio $1/2$, the radius of the nine-point circle is half of the radius of the circumcircle of the original triangle. Since the nine-point circle passes through D, the midpoint of BC, we assert that if N lies on the incircle of ABC then it has to coincide with E. Hence $ND = 2r$. But, if R is the circumradius of

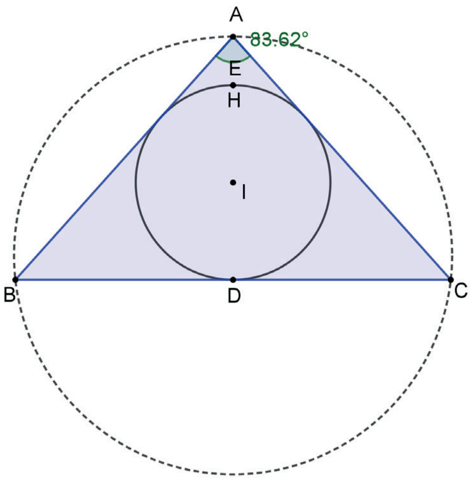

Figure 4

ABC then $ND = R/2$. Therefore $R = 4r$. Now

$$\frac{R}{r} = \frac{x^2y(2x+y)}{8[ABC]^2} = \frac{2x^2y(2x+y)}{y^2(4x^2-y^2)} = \frac{2x^2}{y(2x-y)},$$

whence

$$\frac{2x^2}{y(2x-y)} = 4$$

implying that

$$(x/y)^2 - 4(x/y) + 2 = 0.$$

Thus $x/y = 2 \pm \sqrt{2}$ and $\sin(A/2) = y/2x = (2 \pm \sqrt{2})/4$. The corresponding values of $\measuredangle BAC$ are 117° and 17°, approximately. See Figure 5. (The figure for the other possibility has not been drawn as there is a difficulty with the scale.)

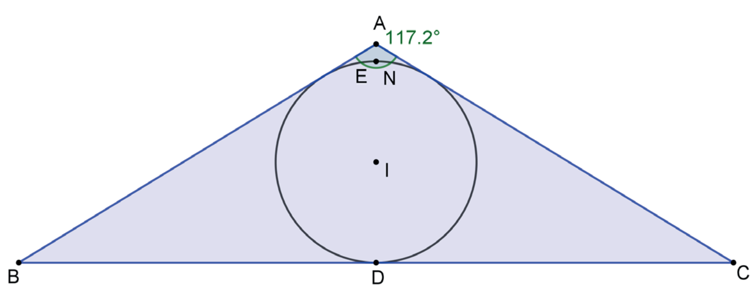

Figure 5

PRITHWIJIT DE is the National Coordinator of the Mathematical Olympiad Programme of the Government of India. He is an Associate Professor at the Homi Bhabha Centre for Science Education (HBCSE), TIFR, Mumbai. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. He may be contacted at de.prithwijit@gmail.com.