Searching for a 3rd Order Magic Square of Squares

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Third order magic squares, i.e., of order 3×3, are the simplest kind of magic squares. Nevertheless, there are a few unsolved problems concerning such objects. For example we may ask:

Problem. Does there exist a 3×3 magic square composed only of square numbers?

This problem remains unsolved. As of today, no such magic square has been found. Nor has it been proven that no such magic square exists.

Recall that a magic square is a square array of distinct positive integers, arranged so that the sum of the numbers is the same for each row, each column, and the two main diagonals. This common sum is the *magic sum* of the square.

The structure of 3×3 magic squares

For a 3×3 magic square, this means that 8 sums need to be equal to one another. The most familiar such square is the following:

4	9	2
3	5	7
8	1	6

(1)

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Here the magic sum of the square is 15:

$$4+9+2=3+5+7=8+1+6=4+3+8=9+5+1=2+7+6=4+5+6=2+5+8=15.$$

It is known that for any 3×3 magic square, if the magic sum is S and the entry in the central square is a, then

$$S=3a. \tag{2}$$

Readers who are unfamiliar with this result may enjoy trying to prove it.

An immediate implication of (2) is that with each 3×3 magic square, we can associate 4 distinct arithmetic progressions (APs), each of which has a as the central number.

In this article, we propose to find a 3×3 square of perfect squares in which 7 out of the 8 sums are equal to one another. This would be 'close' to being a magic square.

Let the central number of the square be a. From the observation made above about APs, it follows that the magic square must have the following form:

$a-b$	$a+b+c$	$a-c$
$a+b-c$	a	$a-b+c$
$a+c$	$a-b-c$	$a+b$

(3)

Observe the 4 APs:

$$\{a-b,a,a+b\},\quad \{a-c,a,a+c\},\quad \{a-b-c,a,a+b+c\},\quad \{a+b-c,a,a-b+c\}. \tag{4}$$

We can view the 9 elements of the array as made up of 3 separate, disjoint APs of 3 terms each:

$$\left.\begin{aligned}&\{a-b-c,\ a-b,\ a-b+c\},\\&\{a-c,\ a,\ a+c\},\\&\{a+b-c,\ a+b,\ a+b+c\}.\end{aligned}\right\} \tag{5}$$

Observe that each AP has common difference c. This observation is crucial.

To find a magic square of the stated kind, we must find 3 different APs of squares sharing the same common difference. This seems a daunting task at the outset. Let us see where the algebra will lead us.

Looking for 3-term APs composed of perfect squares

If the perfect squares u^2, w^2, v^2 are in AP, then we have

$$u^2+v^2=2w^2 \tag{6}$$

$$\iff 2u^2+2v^2=4w^2$$

$$\iff (u+v)^2+(u-v)^2=(2w)^2. \tag{7}$$

Hence the triple $(u+v, u-v, 2w)$ *is Pythagorean.* (Note, however, that it is not primitive; all the numbers in the triple are even.)

We now invoke a well-known way of generating Pythagorean triples: by using the formula

$$\left(m^2 - n^2, 2mn, m^2 + n^2\right), \tag{8}$$

with $m > n$. This formula generates infinitely many Pythagorean triples.

Equating the triple $(u + v, u - v, 2w)$ with the above triple, i.e., setting

$$\left.\begin{aligned} u + v &= m^2 - n^2, \\ u - v &= 2mn, \\ 2w &= m^2 + n^2, \end{aligned}\right\} \tag{9}$$

and solving for u, v, w, we get

$$u = \frac{m^2 - n^2 + 2mn}{2}, \qquad v = \frac{m^2 - n^2 - 2mn}{2}, \qquad w = \frac{m^2 + n^2}{2}. \tag{10}$$

Evidently, m, n must be both odd or both even for u, v, w to be integers.

Using (10), we can generate an unlimited supply of 3-term APs of perfect squares. Thus we have:

m, n	u, v, w	u^2, v^2, w^2	AP	Common difference
3, 1	7, 1, 5	49, 1, 25	1, 25, 49	24
5, 1	17, 7, 13	289, 49, 169	49, 169, 289	120
7, 1	31, 17, 25	961, 289, 625	289, 625, 961	336
9, 1	49, 31, 41	2401, 961, 1681	961, 1681, 2401	720
10, 4	82, 2, 58	6724, 4, 3364	4, 3364, 6724	3360
14, 4	146, 34, 106	21316, 1156, 11236	1156, 11236, 21316	10080

Looking for the required magic square

We now use the above data to generate a 3×3 square with the required property. Our strategy will be to generate a large number of 3-term APs of perfect squares and to look for common differences that are repeated 3 or more times. We find the following common differences repeated multiple times:

$$3360,\ 13440,\ 43680,\ 53760,\ 127680,\ 174720,\ 215040,\ \ldots. \tag{11}$$

Three APs for which the common difference is 3360 are the following:

$$\{2^2,\ 58^2,\ 82^2\}, \qquad \{46^2,\ 74^2,\ 94^2\}, \qquad \{97^2,\ 113^2,\ 127^2\}. \tag{12}$$

The 3×3 square that can be formed using these APs is:

46^2	127^2	58^2
82^2	74^2	97^2
113^2	2^2	94^2

Every sum is equal to 21609, except for one diagonal which adds to 16428. That's close!

Here are 3 other such squares:

103^2	446^2	218^2
302^2	233^2	334^2
394^2	62^2	313^2

2729^2	6271^2	1418^2
1838^2	2969^2	6049^2
6161^2	802^2	3191^2

4324^2	6382^2	2836^2
3676^2	4916^2	5458^2
5938^2	1604^2	5444^2

In all of them, the row sums, column sums and diagonal sums are all equal — except for one sum!

References

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