Repunit Primes

SHAILESH SHIRALI

A *repunit* is a natural number all of whose digits are 1; see [2]. (Here, numbers are expressed in base ten.) Elsewhere in this issue of the magazine, Sasikumar proves two properties of such numbers. In this short note we consider the question of which repunits are prime numbers.

Definition. For any natural number n, we define R_n to be the repunit with n digits:

$$R_n = \underbrace{111\ldots11}_{n\ \text{1's}}. \tag{1}$$

A question which arises naturally when we define these numbers is the following:

Problem 1. For which positive integers n is R_n prime?

The first such prime is $R_2 = 11$, but the supply seems quite limited after that:

- $R_3 = 111 = 3 \times 37$ is not prime.
- $R_4 = 1111 = 11 \times 101$ is not prime.
- $R_5 = 11111 = 41 \times 271$ is not prime.
- $R_6 = 111111 = 11 \times 10101 = 3 \times 7 \times 11 \times 13 \times 37$ is not prime.
- $R_7 = 1111111 = 239 \times 4649$ is not prime.

The following should be clear:

Proposition 1. *If n is even, then $11 \mid R_n$.*

It follows that if $n > 2$ and is even, then R_n is composite.

More generally we have the following:

Proposition 2. *If n is composite, then R_n is composite.*

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Indeed, if $a \mid n$ where $a > 1$, then $R_a \mid R_n$. (Here the notation $u \mid v$ means that u is a divisor of v.) This can be seen by direct division. For example:

$$\begin{aligned} R_4 &= R_2 \times 101, \\ R_6 &= R_2 \times 10101, \\ R_8 &= R_2 \times 1010101, \\ R_9 &= R_3 \times 1001001, \\ R_{15} &= R_3 \times 1001001001001, \\ R_{15} &= R_5 \times 10000100001, \end{aligned}$$

and so on; the pattern should be clear. For a formal proof we note that for integers $m, k > 1$,

$$m - 1 \mid m^k - 1. \qquad (2)$$

Indeed, the quotient in the division $(m^k - 1) \div (m - 1)$ is

$$\frac{m^k - 1}{m - 1} = m^{k-1} + m^{k-2} + \cdots + m + 1.$$

In particular, we have:

$$10^a - 1 \mid 10^{ab} - 1. \qquad (3)$$

This implies that $9 \times R_a \mid 9 \times R_{ab}$ and therefore that $R_a \mid R_{ab}$.

This proves that if n is composite, then so is R_n.

The contrapositive of Proposition 2 is the following:

Proposition 3. *If R_n is prime, then n is prime.*

The converse of this proposition is not true; e.g., R_5 and R_7 are composite (as may be seen from the factorizations given above). Nor are R_{11}, R_{13} and R_{17}:

$$\begin{aligned} R_{11} &= 21649 \times 513239, \\ R_{13} &= 53 \times 79 \times 265371653, \\ R_{17} &= 2071723 \times 5363222357. \end{aligned}$$

However, R_{19} is prime.

So we ask again: for which primes n is R_n prime? After R_{19} the next prime is encountered quickly; we find that R_{23} is prime. But following that we have a very long stretch of composite numbers. The next prime number in the list is R_{319}, and after that it is R_{1031}. A strange pattern!

Clearly, there is some mystery here. It is conjectured that there are infinitely many repunit primes. However, the question remains open. See [1] for the list of known primes p for which R_p is prime.

References

1. The Online Encyclopedia of Integer Sequences, "Sequence A004023" from https://oeis.org/A004023
2. Wikipedia, "https://en.wikipedia.org/wiki/Repunit" from https://en.wikipedia.org/wiki/Repunit

SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.