Two Properties of Natural Numbers all of whose Digits are 1

SASIKUMAR K

In this short article we study two properties of natural numbers all of whose digits (when the numbers are expressed in base 10) are 1. (Such numbers, in which the digit 1 is repeated several times, are also known as *repunits*.)

Proposition 1. *The number* $111\ldots11$ *with '1' repeated* n *times cannot be a perfect square for any natural number* $n>1$.

Proof of proposition 1. We need not consider the case $n=2$ as we know that 11 is not a perfect square. Hence we assume that $n>2$.

Let $x=111\ldots11$ with '1' repeated $n>2$ times. Suppose that x is a perfect square, say

$$x=(2m+1)^2 \qquad \text{(for some integer } m>0). \tag{1}$$

Then we have:

$$\begin{aligned} 10^{n-1}+10^{n-2}+\cdots+10+1 &= 4m^2+4m+1,\\ \therefore\ 10^{n-1}+10^{n-2}+\cdots+10 &= 4m^2+4m,\\ \therefore\ 5\left(10^{n-2}+\cdots+10+1\right) &= 2m(m+1). \end{aligned} \tag{2}$$

In (2), observe that the quantity on the left side is odd (since $n>2$), whereas the quantity on the right side is even. Therefore, equality (2) cannot hold. It follows that our assumption that x is a perfect square is wrong.

Hence the number $111\ldots11$ with '1' repeated $n>1$ times cannot be a perfect square for any natural number $n>1$. □

Keywords: Repunit, natural number, divisible

Remark. Proposition 1 can also be established by looking at the remainder in the division

$$\underbrace{111\ldots11}_{n \text{ times}} \div 4.$$

If $n > 1$, the remainder will always be 3, since $111\ldots11 = 11 +$ a multiple of 100, and 11 leaves remainder 3 under division by 4.

On the other hand, no square leaves remainder 3 under division by 4. Hence the conclusion. □

Proposition 2. *The number* $\underbrace{111\ldots11}_{n \text{ times}}$ *is divisible by n if n is a power of 3.*

Proof of proposition 2. We have:

$$\underbrace{111\ldots11}_{n \text{ times}} = 10^{n-1}+10^{n-2}+\cdots+10+1 = \frac{10^n - 1}{9}, \qquad (3)$$

by summing the geometric progression.

Let $n = 3^k$ where $k \in \mathbb{N}$. We claim that

$$\frac{10^{3^k} - 1}{9 \cdot 3^k} \text{ is a positive integer for every } k \in \mathbb{N}. \qquad (4)$$

In other words:

$$10^{3^k} - 1 \text{ is divisible by } 3^{k+2} \text{ for every } k \in \mathbb{N}. \quad (5)$$

We shall use the principle of mathematical induction to establish this.

The proposition is true for $k = 1$ because 999 is divisible by 27 (note that $999 = 27 \times 37$).

Let us assume that (5) is true for some positive integer $k = m$. That is,

$$10^{3^m} - 1 \text{ is divisible by } 3^{m+2}. \qquad (6)$$

Let $10^{3^m} - 1 = a \cdot 3^{m+2}$ for some positive integer a. Then we have:

$$\begin{aligned} 10^{3^{m+1}} - 1 &= \left(10^{3^m}\right)^3 - 1 \\ &= \left(10^{3^m} - 1\right) \cdot \left(10^{2\cdot 3^m} + 10^{3^m} + 1\right) \\ &= a \cdot 3^{m+2} \cdot \left(10^{2\cdot 3^m} + 10^{3^m} + 1\right). \end{aligned} \qquad (7)$$

In (7), consider the quantity $10^{2\cdot 3^m} + 10^{3^m} + 1$. Since 10 is 1 more than a multiple of 3 (i.e., it is of the form $3t + 1$ for some $t \in \mathbb{N}$), every power of 10 will also be of this form. This implies that $10^{2\cdot 3^m} + 10^{3^m} + 1$ is a multiple of 3. It follows that $10^{3^{m+1}} - 1$ is a multiple of 3^{m+3}.

This establishes the inductive step and hence the stated proposition. □

SASIKUMAR K is presently working as a PG Teacher in Mathematics at Jawahar Navodaya Vidyalaya, North Goa. He completed his M Phil under the guidance of Dr.K.S.S. Nambooripad. Earlier, he worked as a maths Olympiad trainer for students of Navodaya Vidyalaya Samiti in Hyderabad. He has research interests in Real Analysis and Commutative Algebra. He may be contacted at 112358.ganitham@gmail.com.