Approximate Angle Constructions

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In an article elsewhere in this issue, the author gives a construction procedure which yields angles close to 20° and 40°. In this article, we explain 'why' the procedure (depicted in Figure 1) gives good results and suggest an alternative procedure that gives better results.

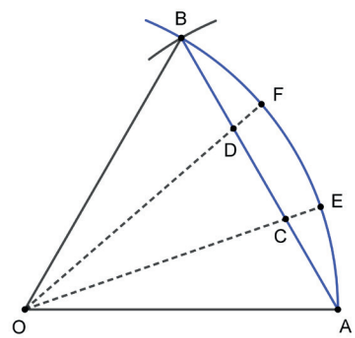

Figure 1. Construct $\angle AOB = 60°$ in the usual manner. Join AB. Trisect AB at points C and D (so $AC = CD = DB$). It will be found now that $\angle COA \approx 20°$ and $\angle DOA \approx 40°$. The approximation is reasonably close, with an error of 4.5%.

To check how close, we use coordinates. Assign coordinates so that $O = (0, 0)$ and $A = (2, 0)$. Then we have $B = 2 \cdot (\cos 60°, \sin 60°) = \left(1, \sqrt{3}\right)$. (Here we use the same symbol for the name of the point and the ordered pair giving the coordinates of the point.) Therefore the coordinates of C and D are given by:

$$C = \frac{2 \cdot A + 1 \cdot B}{3} = \frac{1}{3}\left(5, \sqrt{3}\right),$$
$$D = \frac{1 \cdot A + 2 \cdot B}{3} = \frac{1}{3}\left(4, 2\sqrt{3}\right).$$

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This implies that

$$\tan \measuredangle COA = \frac{\sqrt{3}}{5}, \quad \therefore \measuredangle COA = \arctan\left(\frac{\sqrt{3}}{5}\right) \approx 19.11^\circ.$$

Similarly we find

$$\measuredangle DOA = \arctan\left(\frac{\sqrt{3}}{2}\right) \approx 40.89^\circ.$$

Another procedure that gives better results. Here is a small tweak on the above procedure.

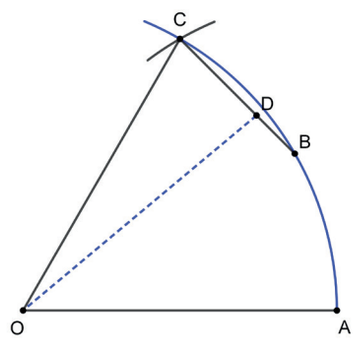

Figure 2. Construct $\measuredangle AOC = 60^\circ$ and $\measuredangle AOB = 30^\circ$. Join BC. Trisect segment BC at point D (so $BD = BC/3$). It is found that $\measuredangle DOA \approx 40^\circ$ and $\measuredangle COD \approx 20^\circ$.

We may analyse the procedure just as we did earlier. Let $O = (0, 0)$ and $A = (2, 0)$. Then we have $B = (\sqrt{3}, 1)$ and $C = (1, \sqrt{3})$. Hence

$$D = \frac{2B + C}{3} = \frac{1}{3}(2\sqrt{3} + 1, 2 + \sqrt{3}).$$

Therefore

$$\tan \measuredangle DOA = \frac{2 + \sqrt{3}}{2\sqrt{3} + 1} = \frac{1}{11}(4 + 3\sqrt{3}),$$

giving significantly better results than earlier:

$$\measuredangle DOA \approx 39.896^\circ, \qquad \measuredangle COD \approx 20.104^\circ.$$

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