ClassRoom

Stewart's Theorem for a Right Triangle – A Compact Proof

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Stewart's theorem states the following (see Figure 1).

Let a, b, c be the sides of $\triangle ABC$.

Let D be any point on side AB, and let p be the length of cevian CD. Let the lengths of AD, BD be m, n.

Then $a^2m + b^2n = c\left(p^2 + mn\right)$.

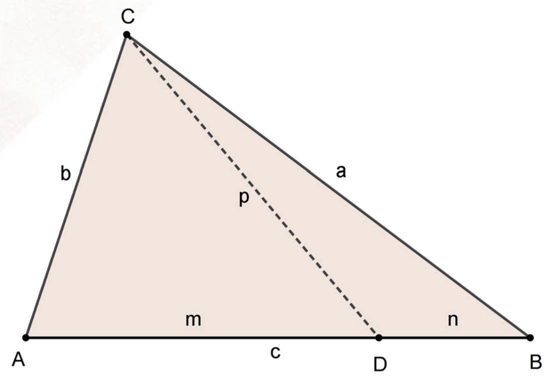


Figure 1

We give below a compact proof for the particular case when ABC is right-angled at C (see Figure 2).

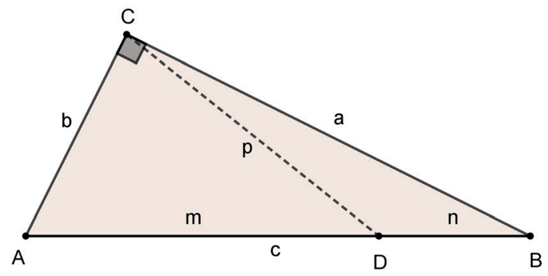
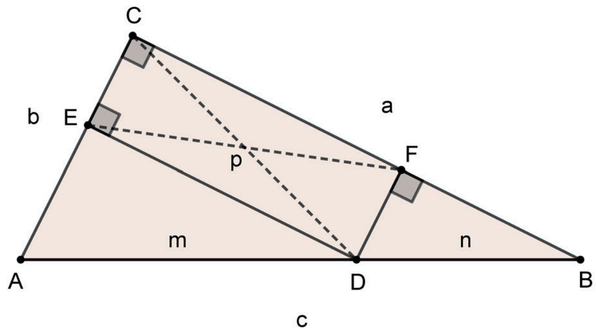


Figure 2

Keywords: Stewart's theorem, right-angled triangle, cevian

We shall prove that $a^2m^2 + b^2n^2 = c^2p^2$ and then show that this is equivalent to Stewart's theorem.

Proof. Construct $DE \perp AC$, $DF \perp CB$ (Figure 3). The proof follows on examining the resulting right-angled triangles.



$$ED = a/c \cdot m$$
$$FD = b/c \cdot n$$
$$EF = CD = p$$
$$p^2 = ED^2 + FD^2 = (a/c)^2m^2 + (b/c)^2n^2$$
$$c^2p^2 = a^2m^2 + b^2n^2$$

Figure 3

We now show that this result is a particular case of Stewart's theorem, $a^2m + b^2n = c(p^2 + mn)$.

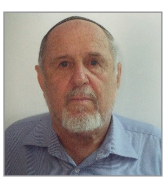
Since $\triangle ABC$ is right angled, $a^2 + b^2 = c^2$. Multiply both sides of the above by $c = m + n$:

$$\begin{aligned} & a^2m + b^2n = c(p^2 + mn) \\ \iff\ & (a^2m + b^2n)(m + n) = c^2(p^2 + mn) \\ \iff\ & a^2m^2 + b^2n^2 + mn(a^2 + b^2) = c^2p^2 + c^2mn \\ \iff\ & c^2p^2 = a^2m^2 + b^2n^2, \quad \text{since } a^2 + b^2 = c^2. \end{aligned}$$

Remark. If $m = n = c/2$ (i.e., if CD is a median), we get $c^2p^2 = a^2(c/2)^2 + b^2(c/2)^2 = c^2 \cdot (c/2)^2$ and so $p = c/2$.

References

1. Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited*. Washington, DC: Math. Assoc. Amer., pp. 6, 10, 31, 1967.
2. Altshiller-Court, N. *College Geometry*, 2nd ed., rev. enl. New York: Barnes and Noble, pp. 152-153, 1952.



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