Teaching Mathematics and Learning About Life…

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I have been teaching high school Mathematics for a few years now. I always considered *teaching* as a sacred service, in engaging students, imparting valuable skills and creating enthusiasm about the subject; but over time I had come to realize that teaching students is as much or more of a *learning* opportunity for me. Here are my experiences as a teacher of mathematics and as a student of life. Invaluable in this journey, is the community of adults deeply involved in teaching and enquiry, in Shibumi and in KFI schools, and in particular, Kabir Jaithirtha, who helped me see and appreciate the richness of teaching, mathematics and life.

Teaching versus Learning…

I began volunteering a few years ago to teach mathematics in Shibumi. For the first few weeks, my primary activity was to attend the senior school math classes – actually to just sit and observe the discussions between the students and Kabir. In no time I realized that just sitting and observing was one of the hardest things to do in life. Quite often, I'd grab any opportunity to "teach" something to the students – lecturing on the topics or explaining away the steps, and often offering solutions even before the articulation of any questions! On such occasions, Kabir would generally veer the conversation away, leaving me feeling quite unfulfilled. Looking back now, it feels mildly comical, that I tried hard filling young minds with skills, instead of allowing them to flower naturally. But at that time, I was just too busy teaching mathematics.

In 2014-15, a school cultural program was arranged with the theme of mathematics (called *MathMela* or *Festival of Mathematics*) to let the children explore the beauty and versatility of mathematics.

Keywords: Mathematics and Life, Teaching Mathematics, Learning Mind, Philosophy of Mathematics, Philosophy of Mind, J Krishnamurti

It was a perfect opportunity for me, providing a structure to work with children on mathematical topics. Students picked, based on their interest or curiosity, various projects ranging from posters to programs, from poems to plays, from games to puzzles, and there were many opportunities to work with students on diverse topics. In one of the projects, I was trying hard to convince a student, that with some changes and extensions, the proposed project could become an excellent showcase of graph theory in the *Mela* and a great learning opportunity for him! But even after many sessions of cajoling, he would not budge! Kabir must have observed the struggle, and one afternoon we had a long conversation on what was happening. *Is the purpose of MathMela to showcase good mathematics projects, or helping the student explore mathematics?* he asked. *From what space is my enthusiasm to improve the project coming from? Shouldn't the natural curiosity and rhythm of the student be allowed to flower on its own?* It was a conversation that revealed to me deeply the very essence of the student-teacher relationship, and the purpose of that engagement – whether in *MathMela,* or in classroom, or even otherwise.

I came to the realization that the *primary* objective of the teacher in any student-teacher relationship is to foster a learning mind, and only *secondarily* to impart knowledge. The subject area is only a field in which a journey of exploration and discovery happens! And, from the teacher, it requires tremendous awareness of him/herself and the space he/she operates from, and watching the space that the student is operating from. After that interaction with Kabir, a significant shift happened in my interactions with the students; I started to try spending more time listening than lecturing; I try leaving the students with questions to explore, than with answers to digest. Importantly, I try watching myself to not let my own enthusiasm crowd out the student's own engagement with the subject. Of course, I fail often, only to remind myself of that very first (and the best) lesson on being an effective teacher: *To partake in a process of learning with the student, and in helping the student in a process of discovery.*

One of the outcomes of the *MathMela* experience was an informal and voluntary extra-curricular program that we started for the senior school students – called *MathLab* – to explore the beauty and versatility of Mathematics. The *MathLab* was literally a lab; started with the conviction that one must do something with one's hands to gain a deep understanding of a subject – even for mathematics. The very first explorations were mathematical artifacts, toys or games – the Abacus, Möbius strip, Slide-rule, interlocking metal puzzles, boardgames, etc. – interspersed with discussion on the mathematics behind the artifacts or the process to generate a solution. Over time, we moved on to reading some classical works in mathematics and exploring puzzles (such as Cissa's chessboard, Königsberg bridges, Zeno's paradox, Hilbert's hotel…). The puzzles and paradoxes provided the best means for mathematical explorations: A good puzzle usually has some history, a simple description, and one feels that the solution may be easy to find; yet, quite likely, either a solution doesn't exist or if it does exist it defies intuition. But either way, it generates good discussion, which is the critical ingredient for learning something new.

Does Provability dominate Truth?

Among the first explorations we took up in 2015 was the reading of the very first work in the development of modern Mathematics – *The Elements* by Euclid. We walked through the definitions, axioms and postulates, and worked through the proof procedure of the first few theorems, to understand the axiomatic systems framework on which mathematics developed over centuries.

In *The Elements,* Euclid developed the planar geometry based on just five simple and self-evident axioms and a rigorous rational process; this geometry – the *Euclidean Geometry* – thus developed is valid even today, more than two millennia later. It was a beautiful process to understand that most non-intuitive results (such as, "*The sum of all the angles of a triangle is two right angles")* emerge naturally out of the basic and simple assumptions. So far, good. Then many natural questions arose: Are all such known results thus proved from the axioms *("Yes")*; Are all possible results already known *("No")*; Are all possible results provable *("May be"),* Are all such possible results true *("Hmmm… ")*. The last question of truth is one of the important questions that came up in the *MathLab* discussions… *How does one know if a given mathematical statement is true?* One thinks Mathematical theorems (such as, *"The sum of the angles of a triangle is two right angles"*) are truths. But, is it so?

Putting this question on hold, we moved on to the other flavours of geometries invented in the 19th century, by altering one or more of the original Euclidean axioms, and the resulting geometries – the Riemannian geometry in which *"The sum of the angles of a triangle is greater than two right angles"* and Bolyai-Lobachevskian geometry in which *"The sum of the angles of a triangle is less than two right angles"*. Much to the surprise of the inventors, these geometries turned out to be as consistent and as valid as the Euclidean geometry! The naturally expected question came up in the *MathLab: Which one of these geometries is true?* This turned out to be one of the very clarifying discussions we had had, as it teased out the two important, but different, concepts – *Provability* and *Truth* – but which are used synonymously in many situations. Each one of the flavours of geometries is consistent within itself – hence valid mathematics. Which one of the three statements about the sum of the three angles of a triangle is true, is a naïve question, as each is provable within the respective geometries. Perhaps a more-informed question is, which one of the three statements is relevant for our context. For example, simple high-school problems may need only Euclidean geometry, but modelling the universe accurately may require the Riemannian geometry. So, we choose the appropriate one for a phenomenon or a situation.

This discussion opened up a very important separation of the meanings (in scientific contexts) of *provability* and *truth*. *Provability* is a phenomenon in an axiomatic system – refers to the fact that a statement is derivable from the given axioms using a logical process – and only says that the *statement is proved in that system* – nothing more, but not trivial either! *Truth* is beyond *provability*, perhaps beyond mathematics, and certainly beyond the scope of this article! The mathematical theorems, though called truths, refer only to the fact that they are *provable* within a logical system. In the Sciences, logical axiomatic systems are used to formulate theories for modeling a given phenomenon (say, the planetary motions, the biological evolution, etc.), and the idea of *scientific truth* is tied to its *provability*. Of course, the more accurately a theory predicts a given phenomenon, the more confidence one has on the theory. Mathematics, of course, provides a rigorous rational framework for formulating and analyzing theoretical models, and thus plays an immensely important and successful role in developing a robust science program.

But, the immense success of the rational systems came to be regarded by many as the *best,* and perhaps, the *only* way to *any* truth in *all* areas of exploration. With rationality as the driving force, *provability* seems to have usurped the place of *truth* in human consciousness, even in situations where its role is not appropriate. The success of Mathematics in the realm of the Sciences is a much-debated topic… Wigner's 1960 paper titled *"The Unreasonable Effectiveness of Mathematics in Physical Sciences"* [1] and the slew of follow-up papers that followed, provide a delightful exploration of this very question from different perspectives. This series of papers laid bare in our minds that mathematics is a wonderful framework for modeling a physical phenomenon as a set of axioms and logical process, but the model's power and appropriateness depend on the fidelity of the axioms and the precision of the physical processes. There are vast areas of knowledge where this approach may not be applicable, such as psychological phenomena that give rise to the sense of individuals or culture. What axioms can we rely on for such analysis? Even if such axioms exist, do such psychological phenomena have the internal order that is demanded for such analysis? Are phenomena like *Beauty, Goodness* or *Truth* even model-able in such frameworks? Is rationality the right framework for such explorations? We just let the questions linger in our minds, so as to understand what mathematics or rationality themselves mean.

Hold ideas lightly; they distort actuality...

In 2016-17, groups of students took up for *MathLab* projects something to do with hands – one group explored Tessellation tiles and another group started the designing of a Geodesic structure... Both were geometric, artistic and required as much use of skillful hands as analyzing minds!

The tessellation project ran for a few weeks, with children learning how to identify basic tessellation shapes, and how to create complex tessellation tiles from simple geometric shapes. Enthused by the wonderful tessellation art by the Dutch artist M C Escher [2], the children started creating tiles of complex shapes: winged-horses, lizards, dancing clowns, etc. The mathematics behind the tessellations was fairly well understood, and they created templates meticulously, with the idea that the template can be copied repeatedly on a large canvas to create a large mural. The first few copies of the template on the canvas fitted very nicely with each other. But, as the canvas progressed, the copies would not fit properly… either there was insufficient space to place the template, or too much space! Reshaping the template tile slightly to make it fit only fixed the error temporarily, but the mis-alignment again happened a few tiles later! That was puzzling… there was precise mathematics behind tessellations and meticulous effort was put in making the templates!

In parallel, the geodesic dome team explored the concepts of geodesics, the mathematics and physics behind them, and even visited an existing large-scale dome. Finally, they decided to build a 20-foot half-dome that could be purposed as a play gym for the younger children, once built. Over the next many months, they spent an enormous amount of time and effort in cutting, pounding, shaping and painting the 100+ steel pipes. After the struts were readied, we started assembling the dome top-down, securing first the apex node with a bolt and a nut, and moving down to the nodes in the layers below. Curiously, a problem similar to that we encountered in the tessellation project started appearing in this project as well! The struts, if aligned perfectly in one side of the dome, would go out of alignment wildly in the opposite side of the dome (by several inches!). Dismantling and restarting from the opposite side or from the

bottom nodes did not help, as the mis-alignment only reappeared elsewhere in the dome! We were left wondering if we had made some serious mis-calculations in the strut design or hand-crafting – a dis-heartening proposition after several months of hard work! But our re-check of the design and re-measurement revealed that the struts were done right; yet, they would not assemble smoothly into a dome!

We let the projects simmer for a couple of weeks... The resolution happened quite unexpectedly; on precise remeasuring, we found that some of the struts were found to be off very slightly (couple of millimeters off over a metre), and bent perhaps a degree or two from the exact angle. When the node aligned perfectly on one end of the dome, the small errors in each strut just accumulated across the dome in the other end to large unsurmountable gaps! It occurred to us that if we keep nodes loosely bolted – *but not tightened fully* – then the resulting flexibility of the structure may allow all nodes to be aligned and bolted properly… Amazingly that was all that was needed to fix the problem! The dome stayed wobbly but flexible, allowing us to align every node, and once all bolts and nuts were in, we progressively tightened each of them to get a solid and rigid structure! Once the solution happened here, the same idea helped us to solve the tessellation project mis-alignment as well: The template tile that was created had some tiny unavoidable errors, but these errors kept accumulating as we progressed along the canvas, making either the space too narrow for the template, or too big, leaving wide gaps. All we had to do was to loosen our rigid holding on to the template!

In both the projects, we were holding rigidly on to the template or the strut, and the very rigidity was the source of our problems… and it can be solved only by maintaining a level of flexibility.

For a life of enquiry, it is absolutely necessary for one to have seriousness of intent and internal order, but also freshness in seeing and lightness of living, so as not to be deluded by fixed ideas.

All of life is relationship…

Around the same time (2016-17), a group of senior students were exploring the nature of the numbers (say, *Natural, Integer, Real, Imaginary,* etc.), that form the very basis for vast areas of mathematics[1]. A seemingly self-evident quest, but which started off some deep personal explorations. We discussed some philosophical perspectives on numbers – the *Platonic* view that the numbers exist only in the world of ideals, the *Fictionalisic* view that they are useful but merely fiction, and the *Nominalistic* view that they arise out of our phenomenological experiences before becoming a part of our language and mathematics. Mostly our group tended to be nominalists, that the numbers were defined as abstractions out of our experiences in the phenomenological world (say, the idea of *three* – written as *3* – abstracted from sensual experiences, such as, the three peaks, the three stars in *Orion belt*, etc.). After some initial readings on numbers, the discussions moved on to the axiomatic Number theory – the Peano axioms for Natural Numbers, where a simple definition of numbers (0, 1, 2, 3, …), and a set of simple operations (such as, +/- and */÷), give rise to fantastically complex results and intricate theorems (such as, "*There exist infinitely many prime numbers", "The prime factorization for a given composite number is unique", Fermat's Theorem,* etc.). Most of such theorems have been proven or disproven, many of them are conjectured and unproven, and there must be infinitely many such theorems that are unknown now, and perhaps never to be discovered!

[1] The idea for such discussions started with an earlier conversation during MathMela time with Kabir, when we wondered if any given number (say, 1, 42, $\sqrt{2}$, π or i) has any intrinsic meaning other than its relationship to all the other numbers?

These observations led to a fundamental and tangential question: Are these theorems inventions or discoveries? There were enthusiastic arguments from students supporting each option; I can only recall with wonder when after a while we concluded that these theorems are all defined implicitly by the Peano axioms themselves, or perhaps when the idea of Natural numbers and their operations arose in human consciousness. How else can it be? As we discussed these ideas more, we could sense that while the phenomenological experiences only helped naming a few numbers and extended them indefinitely, it must *be the simple orderly arrangements and the arithmetic relationships defined by operations,* which imbibe them with rich meanings and enable the emergence of beautiful theorems! It was a sort of Copernican inversion that happened in our thinking about numbers and the relationships between them: The theorems are the expressions of the orderly patterns imposed by operations on numbers, just like the beautiful patterns in a kaleidoscope are nothing but simple pieces of coloured trinkets arranged in an orderly manner by their reflections in the mirrors.

Serendipitously, around the same time the geodesic dome project was getting completed…

The half-spherical dome stood about 6-foot tall, and about 20-foot across; and became literally a hang-out place for children – who sit on it to read books or eat snacks, or climb it or swing in it. The steel half bubble was quite a sight with its orderly spatial arrangement of the *struts* (the steel pipes) and the *nodes* (the bolt-and-nut holding all struts converging at a point). Standing in front of it, a natural question came up for discussion: Which one of the two – the *nodes* or the *struts* – is the basis for the shape of the dome? While visually the nodes of a dome are prominent, the physics and the mathematics of the geodesic dome reveal that the placement of the nodes itself is determined by the struts and their lengths. The nodes are just the points of convergence of the struts, and, depending on the numbers, lengths and arrangements of the struts, the nodes may be shifted at will, and the shape of the dome altered from being a simple sphere to a radically different shape! Aren't the nodes of a geodesic dome like the numbers, and the struts like the arithmetic relationships between them? Or, vice versa! And, the complex theorems are the patterns of relationships between struts and the shapes that are possible in a geodesic dome!

When an insight happens in one area, it brings clarity in many other situations as well. Our discussions moved to another fantastic phenomenon – the *Murmuration* of birds [6], in which a flock of birds fly together forming fantastic shapes that dynamically twist and turn like a real live amorphous animal with real sentience! Physically, that animated figure is nothing but a flock of birds flying as a group, with each and every bird flying not in perfect unison but relating to a small set of its neighbours in fairly simplistic ways. Murmuration of birds is beautifully patterned, but defined solely by the relationships between the simple birds of the flock.

Such ideas made me wonder whether the Self itself is an emergent phenomenon. Metaphorically, the Self is a vast geodesic dome, in which each node may be the individual symbols/ideas (about people, objects, events, phenomenon, etc.), relating to each of the other nodes in innumerable ways. Just as the shape of the dome is an expression of all the struts, the Self may be just an expression of all the relationships. Unlike a static geodesic dome, the Self is dynamic with ever-changing relationships, just as the murmuration of the birds, resulting in an illusion of Self that seems independent and alive!

This metaphor may even scale up: Isn't Humanity just a collection of individual selves and their relationships? An individual may be just a node in the geodesic dome of Humanity, with no independent existence other than the myriad relationships that he/she may have with others. The Humanity may be an expression of all relationships, but making prominent the individuals.

Such ideas made me wonder about *Humanity* itself. Metaphorically, if humanity is a vast geodesic dome, then each node is an individual *Self*, relating to each of the other nodes (people, objects, ideas, etc.) in innumerable ways. But, just as in the Geodesic dome, where the nodes are defined by the struts, perhaps an individual *Self* is nothing but the myriad relationships. Just like the flock of birds, Humanity appears to be composed of individual selves, but is only a collection of relationships giving rise even to the individual selves. Perhaps an individual is just like a node in the geodesic dome, a part and parcel of humanity, with no separate existence whatsoever, but appears prominently as a point in space and time. On the other hand, the sense of self in an individual is like the murmuration of the birds – with each idea and its interrelationships with other ideas interacting and reacting, giving rise to and sustain the complex sense of self that seems real and independent.

Mathematics shows how a well-defined order among numbers can bring about such great beauty as theorems. Geodesic domes are artifacts of beauty and utility, brought about by orderly relationships. Murmuration of birds shows how the fluidity of interrelationships can create immense beauty and dynamism to life. Perhaps order in relationships can bring harmony and beauty to the individual and humanity.

Rationality and the Other…

The great success of mathematics in many areas of knowledge had generally led to the conviction that mathematics *expresses only truths* or *is the truth.* Such conviction was expressed variously from the early Greeks (*"The highest form of pure thought is in mathematics"* by Plato), to medieval scholars (*"Mathematics is the language in which God has written the universe"* by Galileo), to modern mathematicians (*"Mathematics… possesses not only truth, but supreme beauty"* by Russell, *"God used beautiful mathematics in creating the world"* by Dirac, and *"An equation… expresses a thought of God"* by Ramanujan).

About a century ago, two great mathematicians – Alfred Whitehead and Bertrand Russell – were working on a logic framework to generate all the truths of a branch of mathematics (Number Theory), symbolically and systematically; their work, the *Principia Mathematica,* is comparable to the Biblical account of mankind's effort to build the *Tower of Babel* to reach the heavens. However, in 1931, an Austrian mathematician, Kurt Gödel, proved in his revolutionary work – the *Incompleteness Theorems* – that the goal to generate all mathematical truths systematically is bound to fail, as any such system will be incomplete. The great irony of Gödel's work is that he used the very same rigorous principles of mathematical logic to prove the incompleteness of any kind of logical system.

In my mind, Gödel's work is a jewel in the crown of mathematics. The logical framework and the Incompleteness Theorem taken together reveal mathematics *as it actually is:* a very powerful logical framework that helps in modeling and studying of many phenomena, but a far cry from being the revealer of all truths. If a rational system cannot assure discovery of *all* truths even with respect to its own self, then how can it be relied on as a means to discover *all* truths? In using rationality as a tool for understanding all phenomena, are we confusing *truth* with *provability?*

Even more importantly, the impeccable internal order of mathematics and how such an order can reveal its own limitation, holds an immense value for humanity. For me, it was a deep insight that revealed a parallel between Mathematics and the life of enquiry. Like mathematics, the internal order is an absolute necessary condition for any enquiry into the nature of truth; and, like mathematics, the rationality may not be sufficient for enquiry into the nature of truth. For a mind that is latched on to rationality – and hence provability – perhaps truth may be unreachable.

I wonder if the mind is in total internal order to observe *what is,* and is aware of its own structure and its inherent limitations, then perhaps such a mind may be free and pliable enough to move in a totally different dimension, for truth to be. And, the understanding of such a state of mind for a life of enquiry is perhaps the best learning I have gained by teaching mathematics.

References

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KABIR JAITHIRTHA (1949-2018) was an explorer of life throughout his life in the tradition of J Krishnamurti. He was deeply involved in the activities of Krishnamurti Foundation of India (KFI) for nearly four decades, and was a trustee of KFI till the end. Kabir started as a teacher in The Valley School, Bangalore, and later served as the Director of Rajghat Besant School in Varanasi. He was instrumental in starting two independent schools (Centre for Learning in 1990, and Shibumi in 2008), in Bangalore.

Kabir's zest for truth reflected in every aspect of his life – whether engaging a child in a conversation, exploring a subject with a student, enabling a learning environment at school or enquiring truth in dialogue. Our interactions – whether exploring our mutually favourite subject – Mathematics, discussing personal or school issues, or engaging in dialogue – had moved to a life of enquiry, for which I am forever grateful.

SHIBUMI (https://www.shibumi.org.in) is a learning centre for both adults and young people. For interested adults, it offers a space to understand oneself and one's relationship to life in the light of J Krishnamurti's teachings; for the child, it is an open and supportive environment for exploring itself and its relationship to nature and the society, and for developing academic skills. The centre's philosophy is to help the student to learn not only the subject, but to understand the whole activity of learning, and to become a complete human being. Shibumi is located in South India, on the outskirts of Bangalore.