Making the Great Icosahedron

Features

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The great geometer H. S. M. Coxeter wrote in the preface of his book [1], "The chief reason for studying regular polyhedra is still the same as in the time of the Pythagoreans, namely, that their symmetrical shapes appeal to one's artistic sense." Indeed, although the mathematics involved in discovering and classifying various polyhedra [1] may not appeal to everyone, the beauty and symmetry of these objects [2] appeal to young and old alike.

A good classroom activity across the world is to make models of the five Platonic solids. The history of these objects goes back to the ancient Greeks. Famously, Euclid's *Elements* ends by showing that the only solids with congruent regular convex polygons as faces with all vertices of the same type are the five Platonic solids (Wikipedia [6]; see Figure 1 for the definition of a regular polyhedron).

Note that in the previous paragraph, I used the term regular convex polygons, not regular polygons. In Euclidean geometry, a regular polygon is a polygon that has all angles equal and has all sides of the same length. Regular polygons may be either convex or star[1]. A natural question is then, what are the analogs of the Platonic solids if we use star polygons? Kepler, around 1619, noticed that twelve pentagrams can join in pairs along their sides and meet in fives at their vertices to form a solid. See Figure 2. This is a regular solid: all faces are regular polygons (pentagrams) with the same number of faces (five) meeting at each vertex. But it is not *convex*. (A polyhedron is said to be convex if all its diagonals are inside or on its surface.)

Kepler also noticed that the regular pentagrams can join in another way. They can meet at their vertices in threes instead of fives, and they enclose a different solid. This solid is also regular. It is also not

1 See the glossary for an example of a star polygon.

Keywords: Platonic solids, Kepler-Poinsot solids, great icosahedron, hands-on learning, paper models

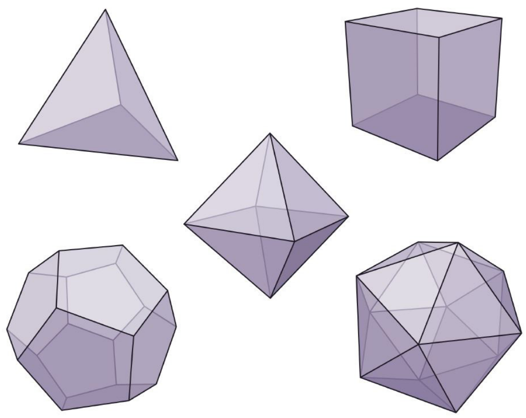


Figure 1. The five Platonic solids. Image from Wikipedia. A Platonic solid is a convex regular polyhedron. Its faces are congruent, convex regular polygons, and the same number of faces meet at each vertex. There are only five such polyhedra.

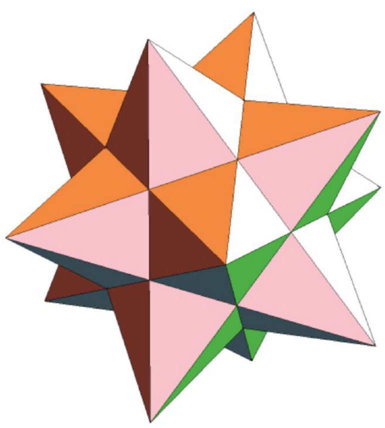
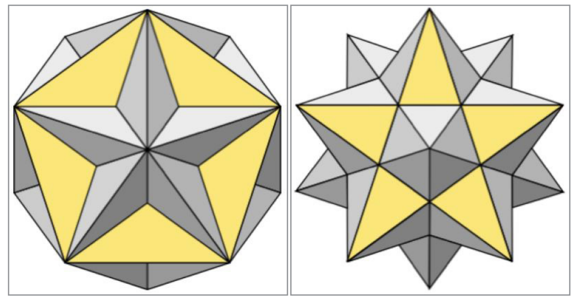
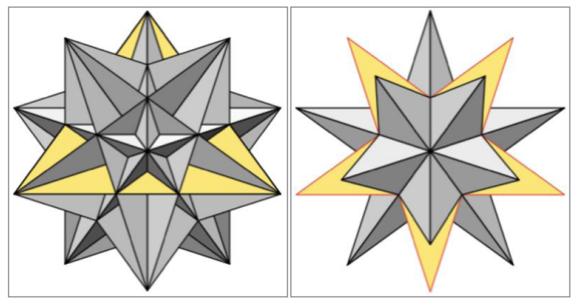


Figure 2. Kepler, around 1619, discovered this solid. It is starlike. Image from Wolfram Demonstrations Project.



Great dodecahedron | Small stellated dodecahedron



Great icosahedron | Great stellated dodecahedron

Figure 3. The four Kepler-Poinsot solids. Image from Wikipedia.

convex. The two Kepler solids are called the small stellated dodecahedron and the great stellated dodecahedron [2]. See Figure 3. The reason for the names will become clear shortly.

One can say that Kepler discovered his two new solids by discarding the ancient Greek concern for convexity. Are there more? It turns out there are two more nonconvex regular solids. These are called Poinsot solids, named after Louis Poinsot, who discovered them in 1809. One of them is a version of the dodecahedron, called the great dodecahedron. The faces are simply twelve pentagons, but the pentagons now intersect each other. The second one is a version of the icosahedron, called the great icosahedron. It is made of twenty intersecting equilateral triangles. The triangles meet along edges at twelve corners as in the icosahedron. See Figure 3.

The Kepler-Poinsot and Platonic solids are members of a bigger class called uniform polyhedra. It is a common mathematical hobby to make models of uniform polyhedra. It is also one of my hobbies. A uniform polyhedron has regular polygons as faces, and all its vertices are equivalent. The faces and vertices need not be convex, as many uniform polyhedra are non-convex, sometimes called star polyhedra because of their star-like appearance. A uniform polyhedron may be regular if all its faces and edges are alike, quasi-regular if all its edges but not faces are alike, or semi-regular if neither edges nor faces are alike. If we do not count prisms and anti-prisms[2], then there are exactly 75 uniform polyhedra [5].

2 See glossary for definitions of a prism and an antiprism.

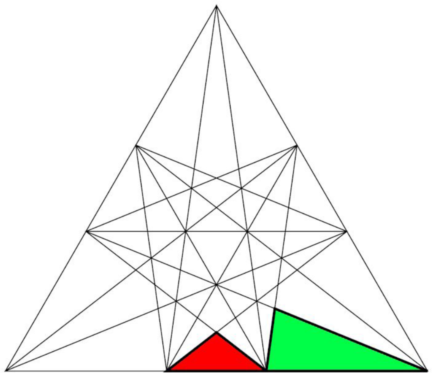


Figure 4. The large equilateral triangle is a facial plane of the great icosahedron. On each side of the triangle we locate two points dividing the sides of the triangle in the golden ratio: $\tau : 1 : \tau$ where $\tau = \frac{\sqrt{5}+1}{2}$. The resulting red and green triangles combined in a fan-shaped pattern as shown in Figure 5 give the basic module for the model.

Father Magnus J. Wenninger (1919-2017) devoted much of his working life to making polyhedra models. The story goes that after he made 65 out of the 75 uniform polyhedra to display in his classroom, he contacted Cambridge University Press to see if there was any interest in a book on polyhedra models. The publishers indicated an interest only if he built all 75.

Wenninger did complete the models. To make the last 10 models, he needed the help of a computer. The difficulty lies in the exact measurements for lengths of the edges and shapes of the faces. This was the first time that all of the uniform polyhedra had been made as paper models. This project took nearly ten years, and the book [4], Polyhedron Models, was published by the Cambridge University Press in 1971.

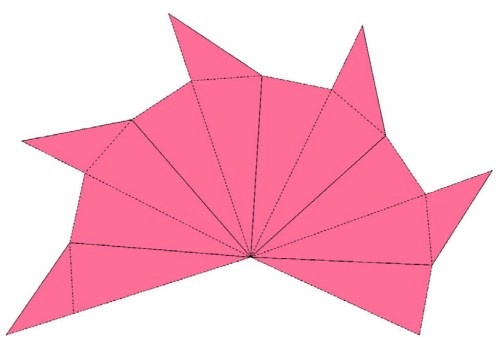


Figure 5. We glue (or put together using a drawing software) 10 of the green triangles and 5 of the red triangles of Figure 4 in a fan-shaped net.

Since then, many enthusiasts have made these models. The Science Museum in London, for example, has a display of all 75 uniform polyhedra models.

During the March 2021 lockdown (working from home), I wanted to make the four Kepler-Poinsot polyhedra. I mentioned one way of thinking about them above. There are several other ways. They can also be obtained by extending the faces of the dodecahedron and icosahedron, a process known as *stellation*. The two Kepler solids and the great dodecahedron can be obtained by stellating the dodecahedron. For this reason, the two Kepler solids are named the small stellated dodecahedron and the great stellated dodecahedron. A concise explanation on stellation can be found in [5] and more details in [1, 2, 4].

All four of the Kepler-Poinsot polyhedra are described inWenninger's book. Making the three stellations of dodecahedron was not difficult. Wenninger's book gives the precise angles for the triangles to be used to make nets: two-dimensional drawings that can be folded into three-dimensional pieces. Various pieces were to be glued together, and we got the models.

The fourth model — the great icosahedron — was not so simple. Wenninger's book tells us to use sketch paper and copy the net from the book. This was unsatisfactory. If I make 120 + 60 triangles in this way, and suppose the angles in the book are slightly off, say, due to the scale of the printing, then the model will not come together. I took a break.

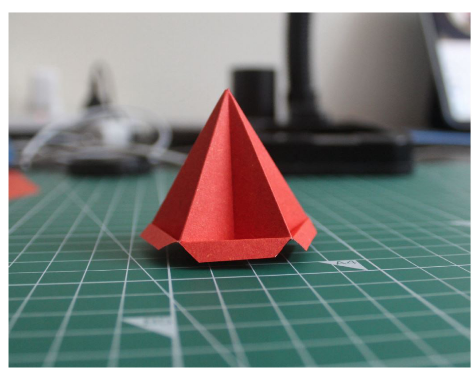


Figure 6. A vertex part for the great icosahedron.

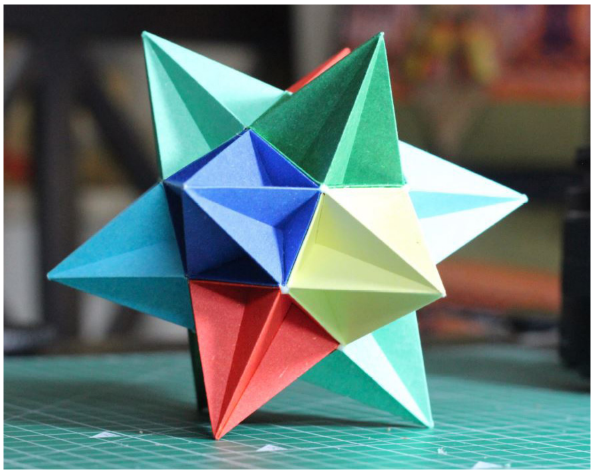


Figure 7. A paper model of the great icosahedron.

Then one day, flipping through the pages, I re-read the preface to the 1978 reprint of the book. It said, "[...] for best results very careful workmanship still demands that you make your own full-scale drawings of all facial planes from which patterns or nets are derived." This is precisely what I did. To make a model of the great icosahedron I began by drawing (in a computer software) a large equilateral triangle: one of the faces of the great icosahedron. On each side of the triangle we locate two points dividing the sides of the triangles (in the computer software) in the golden ratio: $\tau : 1 : \tau$ where $\tau = \frac{\sqrt{5}+1}{2}$. With simple calculations we get the coordinates of the various triangles obtained by connecting those points. For the red triangle of Figure 4 a natural choice (leftmost vertex in the Figure being the origin) turns out to be

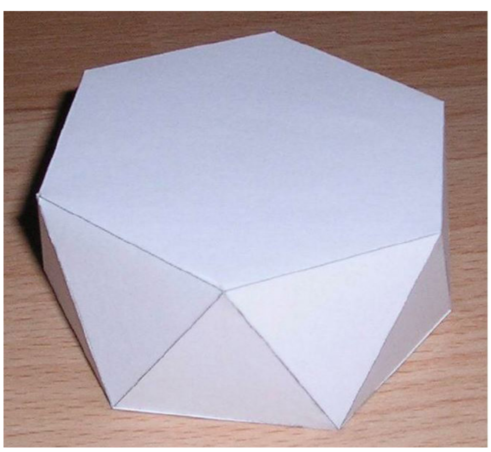


Figure 8. A paper model of a hexagonal antiprism.

$$\left\{\left(\frac{1}{2}\left(1+\sqrt{5}\right),0\right),\left(\frac{1}{2}\left(3+\sqrt{5}\right),0\right),\left(\frac{1}{2}\left(2+\sqrt{5}\right),\frac{\sqrt{3}\left(3+\sqrt{5}\right)}{10+6\sqrt{5}}\right)\right\}, \qquad (1)$$

and for the green triangle it turns out to be

$$\left\{\left(\frac{1}{2}\left(3+\sqrt{5}\right),0\right),\left(\left(2+\sqrt{5}\right),0\right),\left(\frac{5}{4}+\frac{13}{4\sqrt{5}},\frac{\sqrt{3}\left(7+3\sqrt{5}\right)}{20+8\sqrt{5}}\right)\right\}. \qquad (2)$$

We next glue 10 of the green triangles and 5 of the red triangles of figure 4 in a fan-shaped net shown in figure 5.

Next, we must fold the net of figure 5 in an accordion fashion — up and down — up and down. This gives us a vertex part for the great icosahedron, figure 6. Making 12 such vertices and gluing them in a dodecahedron form gives us the final model, figure 7.

It was a lot of fun to make this model and the other Kepler-Poinsot polyhedra models. I explained the construction to school students in our community's summer camp. It was wonderful to see the kids' eyes light up as they understood what was going on.

This model is a delight to hold.

The short article by Wenninger [5] is highly recommended for a quick tour of the world of polyhedra. For a visual account the book by Alan Holden [3] is highly recommended.

Looking at Figure 7, can you make out twenty intersecting equilateral triangles?

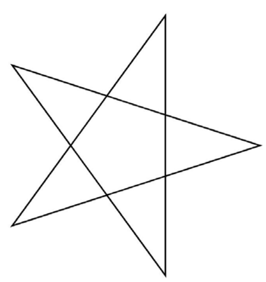


Figure 9. A regular star pentagon has five corner vertices and intersecting edges.

References

[1] H. S. M. Coxeter, *Regular Polytopes*, Dover Publications, 1973.

[2] Peter R. Cromwell, *Polyhedra*, Cambridge University Press, 1999.

[3] Alan Holden, *Shapes, Space and Symmetry,* Dover Publications, April 1992.

[4] Magnus Wenninger, *Polyhedron Models,* Cambridge University Press, 1971.

[5] Magnus Wenninger, "The world of polyhedra", *The Mathematics Teacher,* Vol. 58, No. 3 (March 1965), pp. 244-248.

[6] Wikipedia, "Platonic solid". https://en.wikipedia.org/wiki/Platonic_solid

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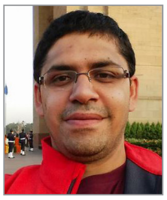

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Glossary

antiprism: An antiprism is a polyhedron whose sides are equilateral triangles, capped at top and bottom by a regular n-sided polygon. See Figure 8 for an example.

facial plane: A face (or facial plane) is a flat surface that forms part of the boundary of a solid object. A three-dimensional solid bounded by faces is a polyhedron.

golden ratio: Two real numbers a > b > 0 are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities, i.e.,

$$\tau = \frac{a}{b} = \frac{a+b}{a}.$$

The Greek letter τ (tau) represents the golden ratio. It is an irrational number, a solution to the quadratic equation $x^2 - x - 1 = 0$, with value $\tau = \frac{1+\sqrt{5}}{2}$.

prism: A prism is a polyhedron whose sides are squares, capped at top and bottom by a regular n-sided polygon.

star polygons: A star polygon is a type of non-convex polygon. See Figure 9 for an example.

stellation: Stellation is the process of extending a polygon in two dimensions or a polyhedron in three dimensions. In three dimensions, starting with a polyhedron, the process extends its faces in a symmetrical way until they meet each other again to form the closed boundary of a new polyhedron.

uniform polyhedra: A uniform polyhedron has regular polygons as faces and is all its vertices are the same.