At Right Angles
A Resource for School Mathematics

SPATIAL THINKING WITH 3-D OBJECTS

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A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

SPATIAL THINKING WITH 3-D OBJECTS

Our brains have adapted over the long period of evolution to a 3-D world and we are able to assess our location and position in space in relation to other objects. When we look at an object depicted on paper (2-D), we automatically construct a 3-D image. We often fail to notice that several actions that we perform in our daily life involve spatial thinking and understanding. As I navigate through the map on my phone or fill up a refrigerator with containers of various sizes, I use spatial understanding. All of us rely on spatial thinking much more than we realise.

A lot of the geometry we study in school is about 2-D shapes and relationships amongst these shapes and their attributes. However, we live in a 3-D world which is even richer in intricate geometric facts and relations. To work with these spaces requires a good understanding of the properties of these objects and ways of visualisation and abstraction. Some concepts we use in 2-D geometry may not apply to 3-D shapes. For example, the shortest path connecting two points on a sphere is not the straight line connecting the two points, and this has implications for air travel.

What is spatial thinking? It is the way the brain processes the position and shape of an object in space. It is through spatial thinking that we understand the location and dimensions of objects, and how different objects relate to each other. It is through such thinking that we construct mental images of objects and visualise them.

The double helix is a famous example of a result of spatial thinking which meets certain requirements. It is a complex 3-D structure with two parallel but displaced spiralling chains.

What does spatial thinking involve? Is it a single skill? Or is it multiple skills? The following are clearly involved:

1. Abstracting the necessary and crucial information (distance, length, coordinates, dimension).
2. Focusing on a certain object embedded in a complicated background and noticing the relationships between the objects and suppressing information not relevant to a task.
3. Representing a design (different views, knowledge of projections, graphs, maps).
4. Scaling an object up or down, or manipulating it in some way.
5. Visualising rotations or symmetry.
6. Visualising an object with a single fold or double fold.
7. Navigating.
8. Memory, synthesis, filling a missing link (closure).
9. Making deductions, evaluating (for actions like taking a detour).

It is thinking which involves the concept of space, tools for representation and a process of reasoning.

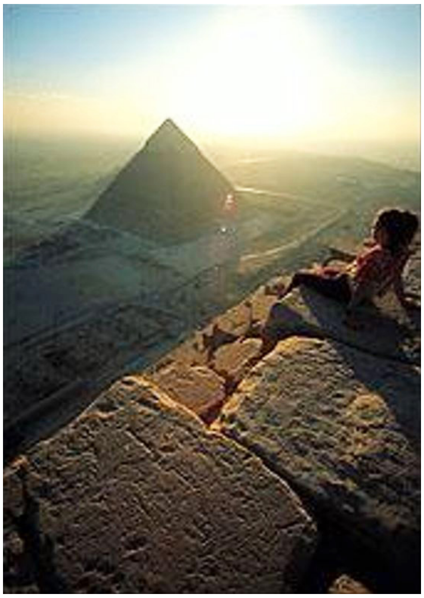

Figure 1

HOW IMPORTANT IS SPATIAL THINKING?

MRI-based research of the brain has revealed that the part of the brain which becomes active during tasks involving spatial thinking is the same as the part used while solving mathematics problems. Spatial thinking can be improved through training and exposure to related tasks.

Many specialists use spatial thinking in their work areas. A civil engineer or an architect performs this feat while designing a building. It is the same capacity that helps a surgeon to navigate the human body and a pilot to fly an aircraft.

While spatial perception and spatial understanding is fundamental to the human thinking process, it poses a challenge in many ways. Our spatial perception can be fooled quite easily, and there are many such puzzles which challenge our perception of objects. Here are two such disturbing examples!

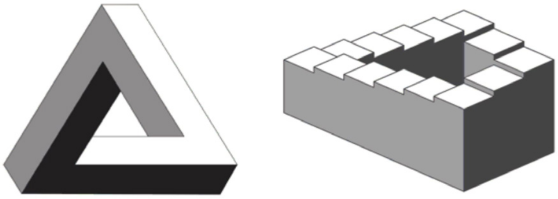

Figure 2

BUILDING SPATIAL THINKING

At primary school level, various activities that involve usage of spatial language, gestures for directional movements, symmetry patterns, reading maps, playing with tangrams are all part of an attempt to build spatial thinking abilities in 2-D space.

Spatial thinking in 3-D must similarly involve handling of 3-D objects, manipulating and studying them. Understanding the location (position) and dimensions (such as the length and size) of 3-D objects, and studying how different objects are related to each other is an important part of this study. It involves building or constructing these objects with blocks or plasticine and clay, studying the nets of such structures, working on paper and pencil tasks, and using geometric software.

While we manage our daily lives with the spatial sense we have developed over time, coping with the complex spatial problems of the world today requires us to use GIS or 'Geographic Information System' technology.

Spatial thinking can be integrated into various subjects as it has relevance in many areas. However, it is good to study it in a 3-D context as a separate unit. Also, regular polyhedra have such beauty to them that it would be a great pity not to make their study a part of the geometry curriculum.

Da Vinci was one person from history who had a tremendous capacity for visualisation. Sculptors like Michelangelo used it when they visualized a future sculpture trapped inside a lump of stone.

Note: Various activities involving usage of 3-D objects can be given to students in a scaffolded manner to build their visualisation skills. The activities incorporate several minor skills like determining and comparing direction, orientation, location, distance, size, colour, shape, and other attributes. Some initial activities can also be used in primary school.

More advanced activities will use major skills like changing perspective (reference frame), changing orientation (mental rotation), transforming shapes, changing size, moving wholes and reconfiguring parts.

Spatial thinking involves visualizing relations, imagining transformations from one scale to another, mentally rotating an object to look at its other sides, creating a new viewing angle or perspective, and remembering images in places and spaces. Spatial thinking also allows us to externalize these operations by creating representations such as a map.

This article focuses on simple mathematical objects in 3-D to develop the capacity to visualize and abstract out their properties.

ACTIVITY 1

Objective: Building a model of a given 3-D construction
Materials: Interlocking cubes, complex structure made with cubes

Let students work in pairs. One student builds a complex 3-D structure. The other student must observe the construction carefully and build a similar one (same size, colour combination and orientation as the one built by the partner).

Can they describe their shape using spatial language (top, left, at right angles, parallel to, …)?

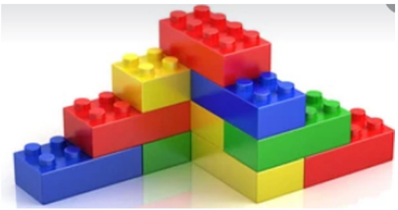

Figure 3

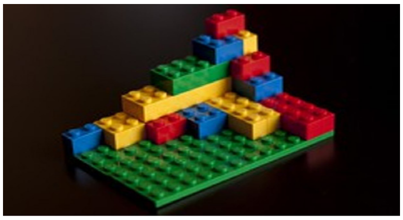

Figure 4

Building complex models helps the students to stay focused on the subtler features of a problem. They need to notice the colour, the length, perceive the parts and the whole and their relationships.

They will also need to notice the angles and the orientation.

ACTIVITY 2

Objective: Building a model using a picture of a 3-D construction

The ability to interpret a drawing, visualise the hidden cubes and reconstruct a model takes the student to the second level of the spatial understanding process.

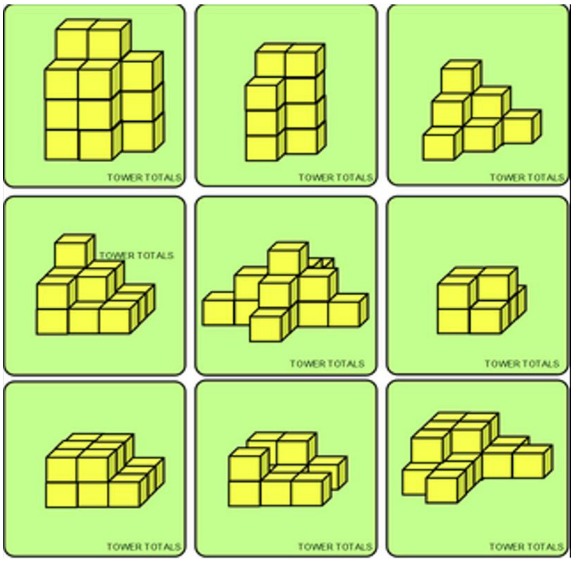

Figure 5

Given some pictures of models, can the students reproduce the models with blocks accurately (assuming no missing pieces)?

Can they figure out the number of blocks they would need for each of these models before building them?

How accurate is their reasoning?

What are the special features of each model?

Does it taper upwards? Does it have symmetry?

Will it look the same if it is rotated?

Figure 6

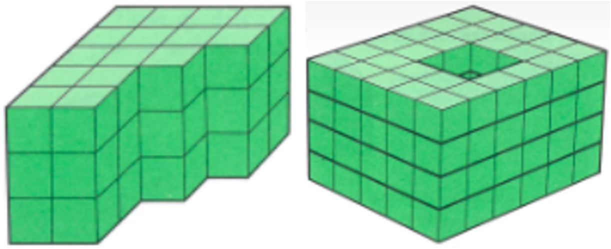

Figure 7

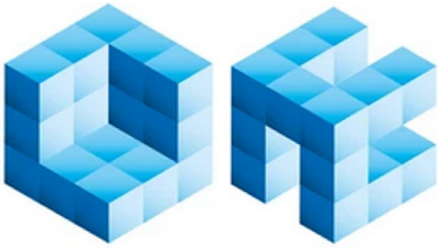

Figure 8

What is the length of the base? What is the breadth of the base? What is the height at its highest point?

How would they figure out the volume of an irregular object?

Can the students compute the volume for these irregular models? What are the different approaches they might use?

ACTIVITY 3

Objective: Building 3-D structures with plasticine/ clay and straw/ toothpicks
Materials: Straw/Toothpicks and Plasticine/Clay, Chart displaying prisms and pyramids with names.

Let students build 3-D structures in pairs to create different 3-D objects. Let them explore the shapes and record the data about vertices, edges and faces.

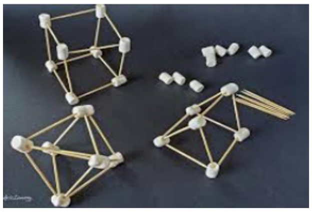

Figure 9

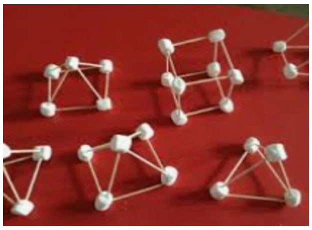

Figure 10

At the end the student pairs can record and consolidate their findings in a table as follows:

Object	Vertices	Edges	Faces
Pyramid with a square base	5	8	5
Triangular prism			
…			

Students should discuss their findings at the end.

How do the numbers of vertices, edges and faces relate to each other? Is there a pattern?

Let students explore ways of figuring out the surface areas of these objects; e.g., using their nets.

How would one find the volumes of these objects? Can the students come up with some ideas?

Project

Pyramids have held a great fascination for many civilisations. Egyptians used them as burial tombs. One such pyramid is the great pyramid of Giza. It measures nearly 480 ft in height and 750 feet at the base and has a slope of 50°. Students can build a scale model of this pyramid and study its features.

If you were to walk around the base, how far would you walk?

What shape would emerge if you sliced a pyramid in half horizontally? Vertically through the apex?

Figure 11

ACTIVITY 4

Objective: Designing the nets for simple 3-D objects
Materials: Prism and pyramid shaped objects

Figure 12

Given the pictures of some 3-D objects, can the students draw the nets?

Here are some possible nets of a square pyramid.

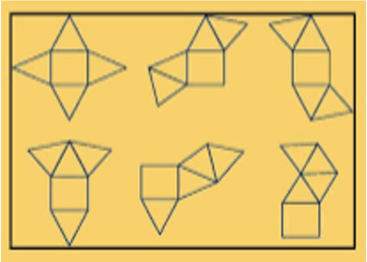

Figure 13

Students will need to clearly know the difference between a pyramid and a prism. A pyramid has one pointed end; slant edges connect it to all the vertices of a polygonal base. The base can be a triangle, producing a *tetrahedron;* it can be a square, giving rise to a *square pyramid;* or it can be a pentagon, or a hexagon, …. There are therefore infinitely many types of pyramids. There is no limit to the number of sides the base can have.

A prism has the same face at both ends. The sides of the face can vary from 3 to any number.

Students tend to have a fixed idea of prisms as they see prisms mainly in the physics lab. However, a prism can have a multi sided base. It can even be L-shaped as shown here.

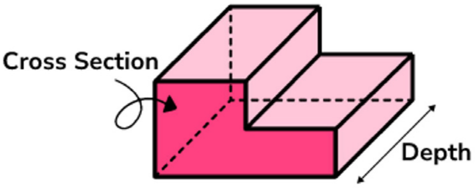

Figure 14

A prism can be cut into layers parallel to one side and all the layers will be exactly the same as shown in the figure.

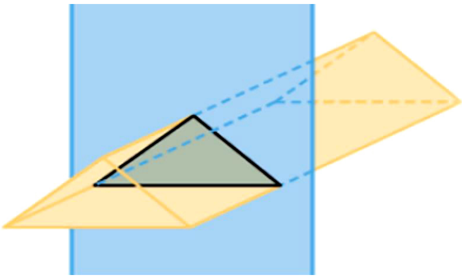

Figure 15

In contrast, a pyramid cannot be cut into layers which are identical to one another. The green square in the figure is not the same as the base of the pyramid.

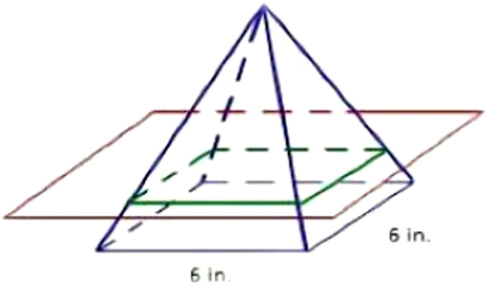

Figure 16

ACTIVITY 5

Objective: Drawing 3-D objects or constructions on isometric/triangular paper

Let the students make drawings (isometric sketches) of different solids on isometric paper.

Here are two such isometric sketches:

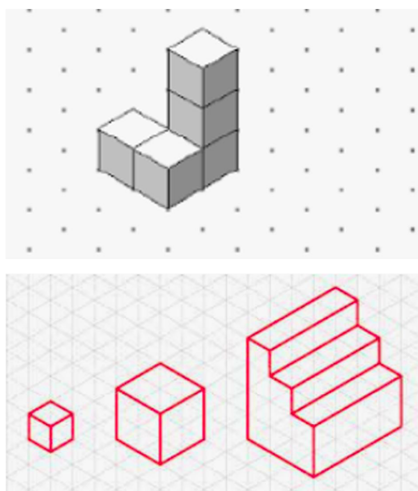

Figure 17

Students can build a variety of different structures and draw them on dot paper. The skill of representing 3-D in 2-D form needs to be built up gradually and students will need hand holding.

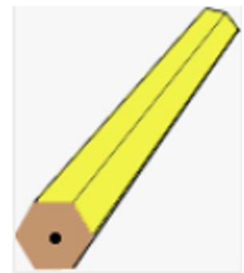

Figure 18

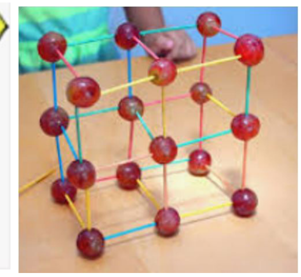

Figure 19

After making the isometric sketch, the students should compare it with the object to check whether the two correspond exactly. A few more complex ones:

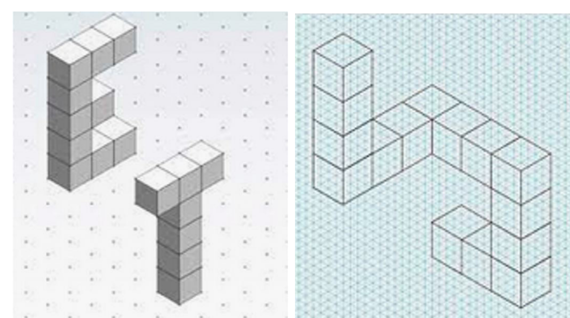

Figure 20

ACTIVITY 6

Objective: Visualising shapes from nets
Materials: Different nets with colour patterns or numbering

If this net is folded, what shape will the object have?

To come up with the answer, students need to form a mental picture of the prism being folded.

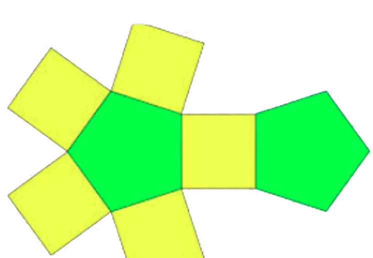

Figure 21

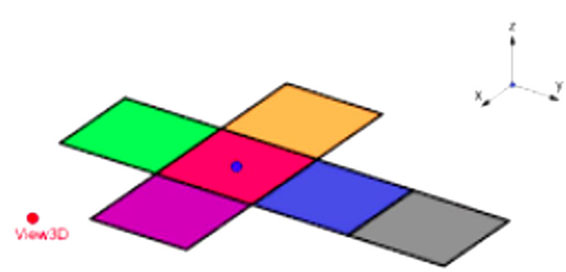

Figure 22

While doing so, they must keep track of the relative positions of the different coloured sides. What shape will be opposite the pentagon?

If the net in Figure 22 is folded, what coloured square will be opposite the pink square? What coloured squares will be adjacent to the green square?

What shape will the net in Figure 23 give rise to?

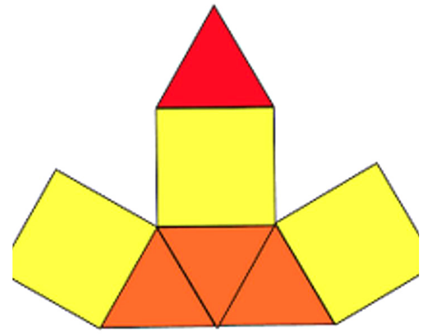

Figure 23

ACTIVITY 7

Objective: Sketching views of simple 3-D structures with two objects
Materials: Couple of blocks arranged touching each other

Drawing the top view, front view and side view is a skill which develops gradually and it is important to start with a few simple objects. (Note: Isometric sketches are made on isometric dot sheets, as shown above, while the views are generally made on square or rectangular grid sheets.)

Here are a few sample drawings.

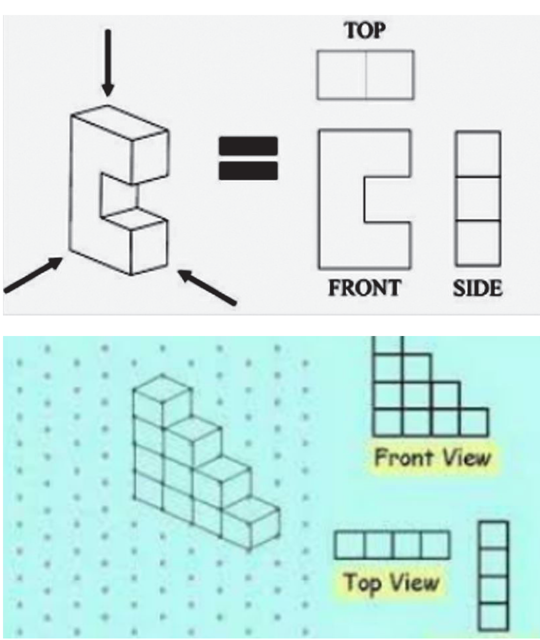

Figure 24

Let the student look at it from the top and draw a top view. It can be followed by a front view and a side view. They can use either isometric or square grid paper to aid in the drawing process.

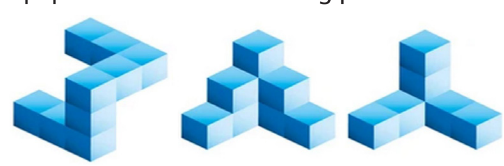

Figure 25

One can increase the complexity of the structure gradually. A few more examples of drawings on dot paper are shown in Figure 26.

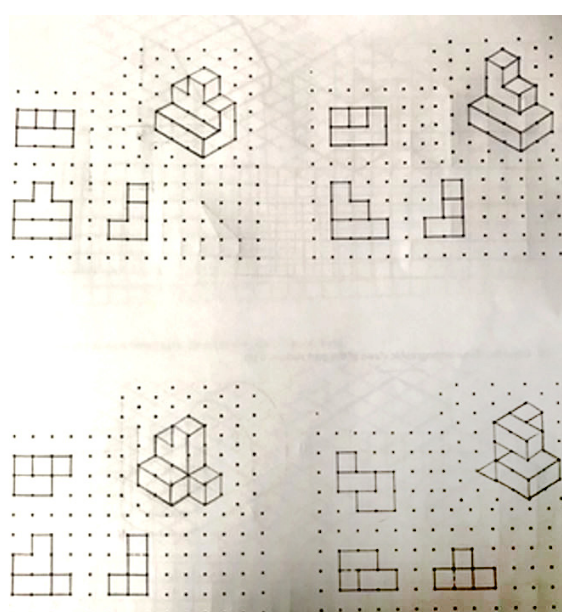

Figure 26

Views

They can study various views of cube structures as shown here.

A Rubik cube will make a very good model for such drawings. Many matching exercises can be created.

Challenge!

Construct a structure made up of 8 identical cubes and having the largest possible surface area.

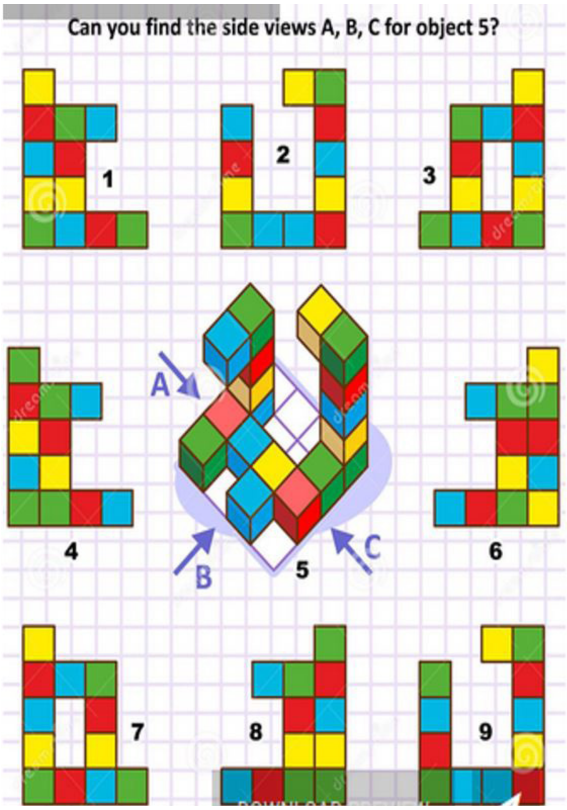

Figure 27

ACTIVITY 8

Objective: Understanding polyhedra and regular polyhedra
Materials: Various mathematical 3-D objects of different sizes or a chart containing pictures of various 3-D mathematical objects
Vocabulary: face, edge, vertex, polygon, polyhedron, regular, convex

Initiate a discussion on sorting the collection into two sets. Students may sort them on the basis of curved surface and plane surface.

Discuss why the word 'polyhedron' is used to refer to an object with polygonal faces.

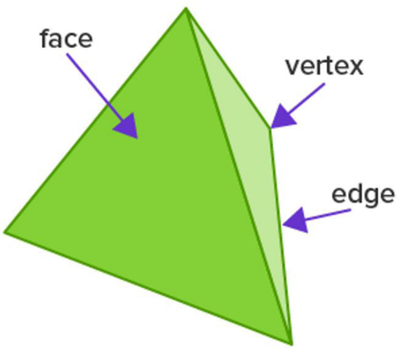

Figure 28

The word 'Poly' refers to *many* and 'hedra' refers to *faces*. Polyhedron means a 'many-faced object'. (The word polygon similarly refers to a many-angled shape.)

Polyhedra are objects which are 3-D, have polygonal flat faces, straight edges and vertices where three or more faces meet. We will consider only *convex polyhedra;* they have no surface indentations or holes.

Cubes and prisms are examples of polyhedra. Cylinders and spheres are not polyhedra.

Can the students now attempt at sorting the polyhedra into different categories? They will notice that in addition to prisms and pyramids, there are other objects which are also polyhedra.

They will see that some of these objects are highly symmetric: they look the same looking down at each face and looking down at each vertex. Their faces are regular, congruent polygons. These are the *regular polyhedra*. They are also known as *platonic solids*.

A polyhedron can fail to be regular in many ways. For example, its faces may not all be congruent copies of one another; rather, the faces may be regular polygons with different numbers of sides (there are many such polyhedra, highly symmetric in appearance). Or the polyhedron may not be convex, i.e., it may have indentations.

(Note: It is not necessary to bring in the notion of polyhedral angles here.)

Discuss, experiment and discover: Raise questions which help students discover that a minimum of 3 faces need to meet at a vertex to form a closed shape.

How many equilateral triangles can meet at a vertex?

Students will notice that if they have six equilateral triangles meeting at a vertex, the triangles will lie flat. Can they justify why it is so?

How many squares can come together at a vertex?

How many regular pentagons can come together at a vertex?

Can regular hexagons come together at a vertex? Why or why not?

This can lead to the discovery that at any vertex of a convex 3-D polyhedron, the sum of all the angles will always be less than 360 degrees. Can they generalise the result?

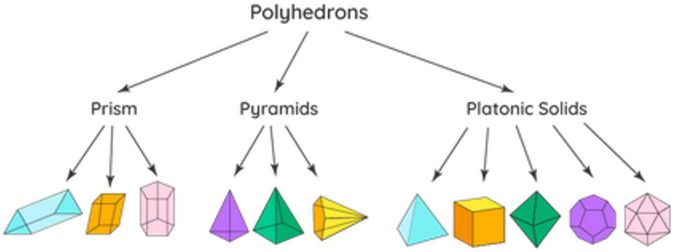

Figure 29

Rotating the objects

The students will need to be given explicit instructions while rotating an object initially. They first start the process by physically rotating them.

At the second stage, they try to rotate the object using their mind's eye.

What would the object look like if it was tilted by 45°? 90°? 120°?

Give students some pictures of pairs of rotated objects.

Ask: Are the two objects different? Or are they actually the same, merely oriented differently?

It's great fun to turn these objects into art pieces by colouring them or drawing patterns on them.

Plane symmetry and rotational symmetry

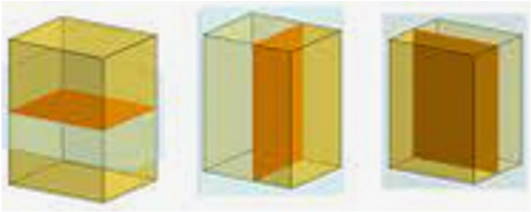

Figure 30

Discuss plane symmetry with examples.

An object has *plane symmetry* if it can be divided into two halves by a plane so that each half is a reflection of the other across the plane.

Such a plane is called a *plane of symmetry*.

A cuboid has 3 such planes of symmetry.

How many planes of symmetry does a cube have?

Discuss *rotational symmetry* with examples.

Practical demonstration is advisable, by piercing holes and passing a straw (or a taut thread, or a wire) through the faces or corners of the paper models.

If a 3-D figure is turned around a fixed line, it is called a *rotation*.

Objects that look the same after a certain amount of rotation are said to have *rotational symmetry*.

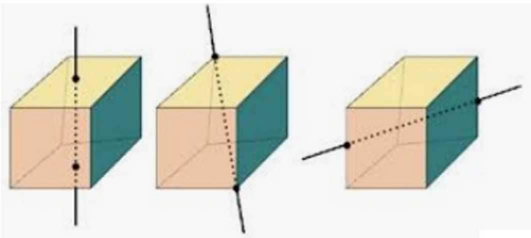

Figure 31

Rotational symmetry is measured in terms of 'order'. When we rotate an object like a cube through 360° about the axis connecting the centres of a pair of opposite faces, the cube fits exactly onto itself four times: after rotations of 90°, 180°, 270° and 360°; so this is called rotational symmetry of order 4. The axis about which it is rotated is called the axis of rotational symmetry. The order of rotational symmetry of a triangular pyramid would be 3, as it fits onto itself after rotations of 120°, 240° and 360°.

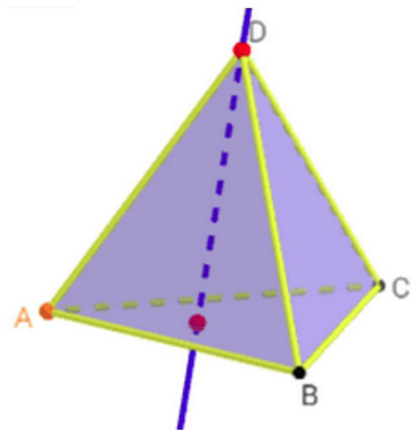

Figure 32

ACTIVITY 9

Objective: Study of regular polyhedra (tetrahedron)
Materials: Straws and plasticine/thread

Thus, the chief reason for studying regular polyhedra is still the same as in the time of the Pythagoreans, namely, that their symmetrical shapes appeal to one's artistic sense.

– H S M Coxeter

What closed structure can be built with equilateral triangles where every vertex has 3 adjacent triangles?

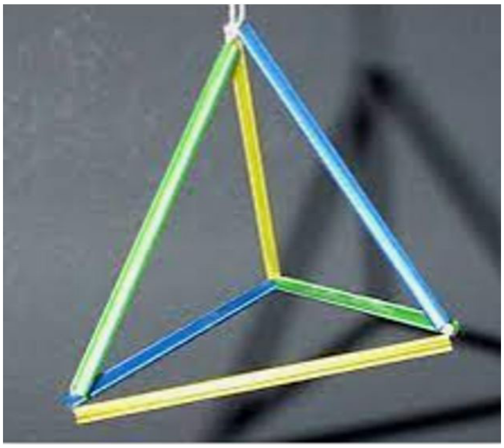

Figure 33

Let students build 3 equilateral triangles about a vertex, using straws.

They will see that they have built a regular tetrahedron (4-faced polyhedron).

Has it formed a closed figure?

Verify that there are 3 triangular faces at each vertex.

What types of symmetry does it have?

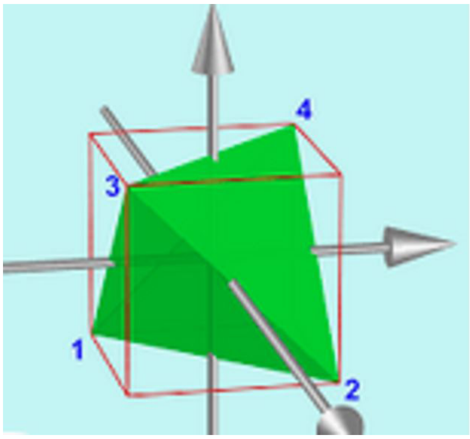

Figure 34

Does it have any plane symmetry? What will the plane pass through? How many such planes can you find in the regular tetrahedron?

Net: Students should be encouraged to design nets for a tetrahedron. They can design and fold the net to make a solid shape.

Pass straws (or wires or taut threads) through holes to check for rotational symmetry.

Does it have any rotational symmetry? Through which points does the axis of symmetry pass?

What is its order?

A **Dihedral meter** (a flexible L-shaped angle measure) can be used to measure dihedral angles (this is the angle at which adjacent faces meet). Students should be able to make such devices for themselves.

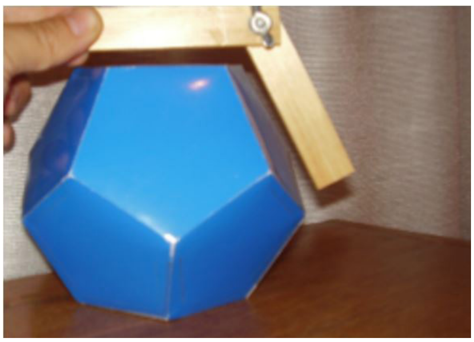

Figure 35

In Figure 35 it is being used to measure the angle between the faces of a dodecahedron.

Students can explore such objects in a variety of different ways, bringing different skills to these explorations:

- They can measure the angles between surfaces using a dihedral meter.
- They can study the relationship of the side to the surface area.
- They can generate different views of the object and sketch them.

There are many resources available online which reinforce their skills and knowledge (e.g., Figure 36).

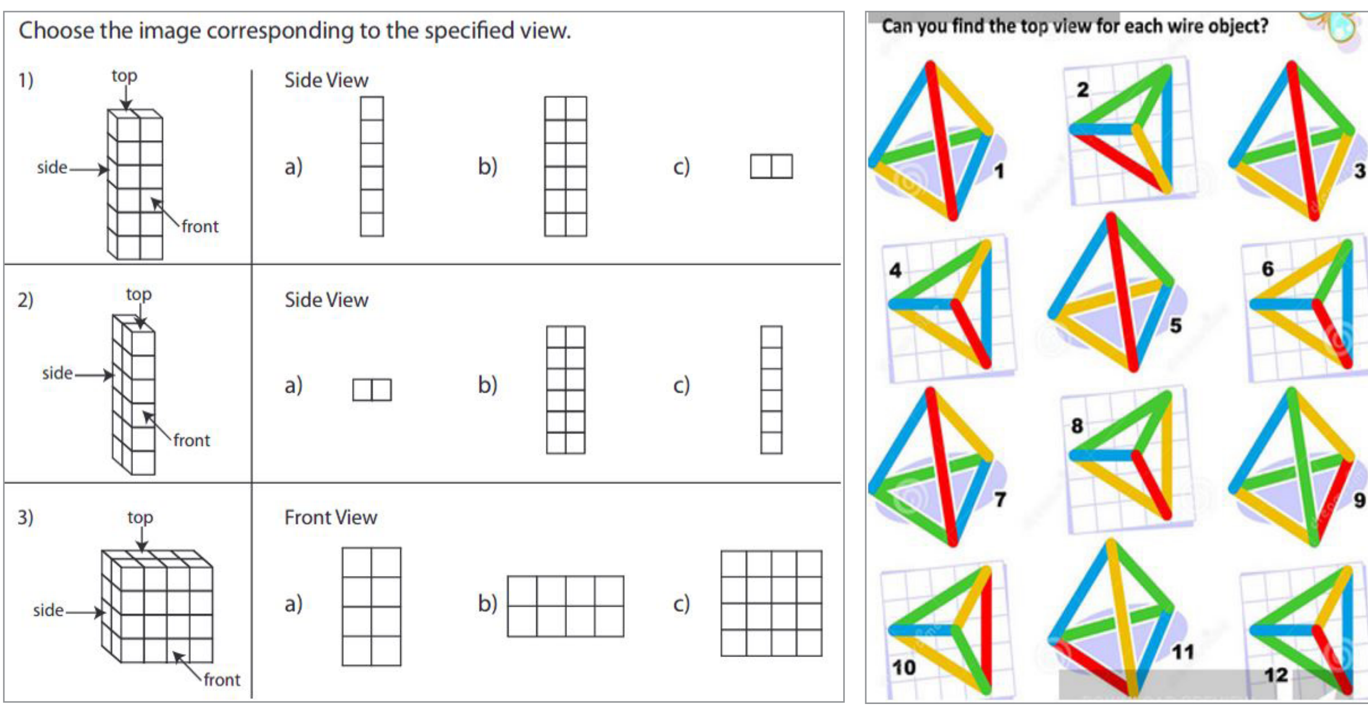

Figure 36

Optional exploration

Cross sections: Students can also explore horizontal or vertical cross sections of the objects

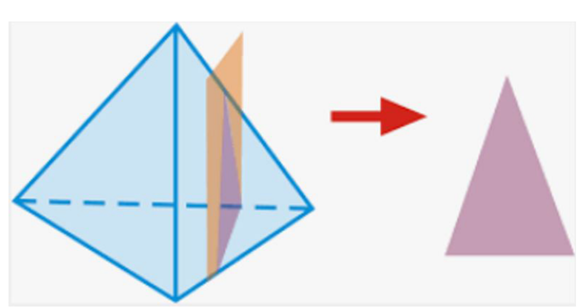

Figure 37

ACTIVITY 10

Objective: Study of regular polyhedra (cube)
Materials: Straws and plasticine/thread

What closed structure can be built using identical squares where every vertex has 3 adjacent squares?

Figure 38

Let students build 3 adjacent squares with straws to form a corner. Let them build more squares with three squares at each vertex.

They have formed a *Hexahedron* (cube).

Does it form a closed figure? Verify that there are 3 square faces at each vertex.

They can explore it and discover the relationship of the side to the surface area and volume of the cube.

They can look for a relationship between the side and any interior diagonal, and between the side and any face diagonal.

Does the cube have any plane symmetry?

Does it have any rotational symmetry? Through which points does the axis of symmetry pass through?

They can bring their understanding of coordinates to a 3-D object and describe various points in terms of coordinates.

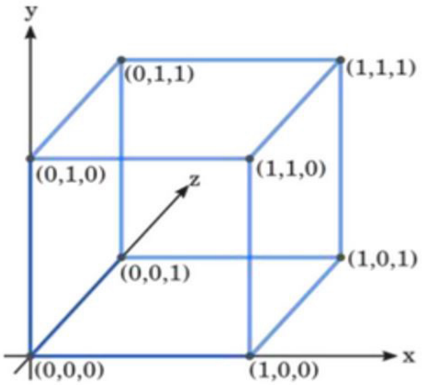

Figure 39

They can also design the net for a cube and fold it to make a solid shape.

ACTIVITY 11

Objective: Study of regular polyhedra (Octahedron)
Materials: Straws and plasticine/thread

What closed structure can be built with equilateral triangles where every vertex has 4 adjacent triangles?

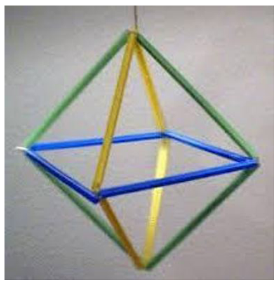

Figure 40

Let students build four equilateral triangles around a vertex. Let them build more triangles with four triangles meeting at each vertex.

A *regular octahedron* has been created. Verify that 4 faces meet at each vertex.

Does it have any plane symmetry?

Does it have any rotational symmetry? Through which points does the axis of symmetry pass through? How many such axes of symmetry does a regular octahedron have?

Cross sections

How would the cross sections (vertical and horizontal) of an octahedron look?

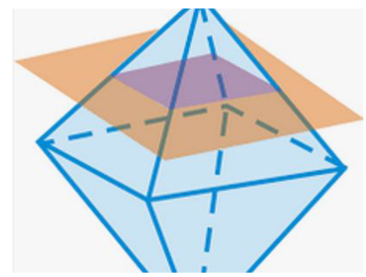

Figure 41

ACTIVITY 12

Objective: Study of regular polyhedra (Icosahedron)
Materials: Straws and plasticine/ thread

What closed structure can be built with equilateral triangles where 5 triangles meet at every vertex?

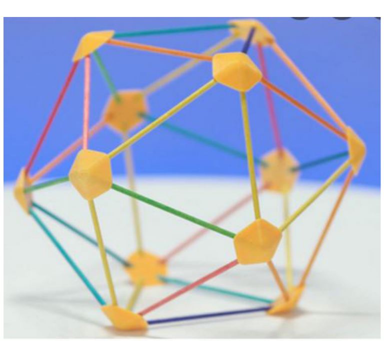

Figure 42

Let students build and join together 5 equilateral triangles, forming a convex shape. In successive steps, at each new vertex they can build 3 more equilateral triangles as shown in Figure 42.

Figure 43

The figure closes to form an *Icosahedron*.

Explore the structure for symmetry.

Building an icosahedron (indeed, building any of the regular polyhedra) using modular origami can be great fun (see Figure 43).

ACTIVITY 13

Objective: Study of regular polyhedra (Dodecahedron)
Materials: Straws and plasticine/thread

What closed structure can be built with regular pentagons where 3 pentagons meet at every vertex?

Let students build a regular pentagon and join together 5 regular pentagons on all its sides.

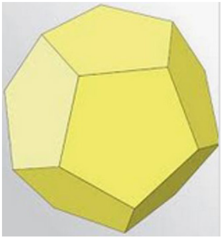

Figure 44

Figure 45

Three pentagons together can be joined to form a polyhedral vertex, but four pentagons together have a vertex angle that adds up to more than 360° making a concave vertex.

In successive steps, at five new vertices they can build five more pentagons as shown in Figures 44-45. The shape closes to form a *Dodecahedron*.

Findings

How many regular polyhedra are possible to build?

We have seen that it is possible to build regular polyhedra with either 3 or 4 or 5 equilateral triangles meeting at each vertex, but not with 6; with 3 squares meeting at each vertex, but not with 4; and with 3 regular pentagons meeting at each vertex, but not with 4 or more.

Is it possible to build a convex shape using only regular hexagons? Three regular hexagons make 360° which then create a flat vertex. Therefore, this is not possible.

Is it possible to build a convex shape using only regular polygons having more than 6 sides? Polygons with more than 6 sides have angles which exceed 120°, so it is not possible to join three of them together at a vertex. Therefore, this too is not possible.

Hence, it is possible to have only 5 regular solids.

Students can now create a table for the five regular polyhedra recording the number of faces, edges and vertices, and describe the face of each of the platonic solids.

Name of Polyhedron	Faces (F) Vertices (V)	Edges (E)	Tetrahe-dron
Tetrahedron	4	4	6

Students can now verify the relationship which they had noticed earlier between the vertices, faces and edges of a Polyhedra structure.

This is Euler's formula: $F + V = E + 2$ where F, V and E stand for the number of faces, vertices and edges of the polyhedron respectively.

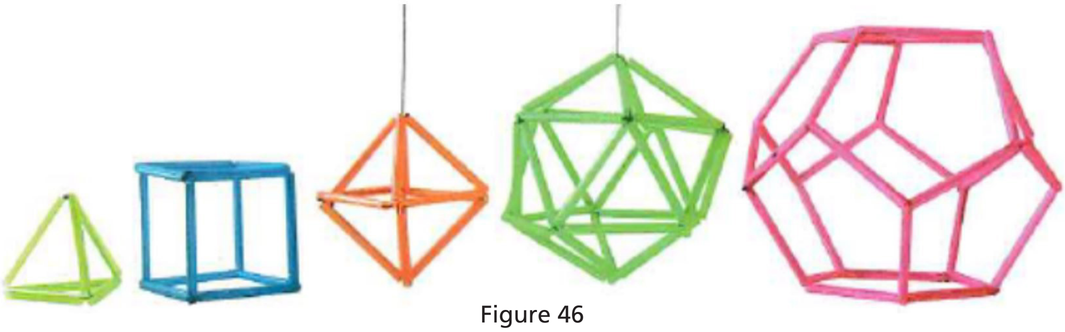

Figure 46

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- Make Space: The Importance of Spatial Thinking for Learning Mathematics · Frontiers for Young Minds (frontiersin.org)
- How to teach ... 3D shapes | Teacher Network | The Guardian

Acknowledgements

I acknowledge with thanks feedback and suggestions received from Ms Swati Sircar.

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