Problem Corner

On a Generalization of a Problem on Factorisation

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In high school mathematics, we come across the topic of factorisation of polynomials with integer or rational coefficients. Usually, we only consider factorisation of polynomials of low degrees. Checking whether or not a polynomial of higher degree such as $X^{37} - 27X^{11} + 3$ is factorisable turns out to be a difficult problem. Although there are some methods available in higher mathematics to deal with such problems, the tools and techniques of high school mathematics seem to be of little use for addressing such problems. In this article, we discuss the factorisation of a particular infinite family of polynomials of arbitrary degree.

Statement of the problem

In [1], it is shown that for any integer $n \geq 1$ and distinct integers $a_1, a_2, \ldots, a_n$, the polynomial

$$(X - a_1)(X - a_2) \cdots (X - a_n) - 1 \tag{1}$$

is not factorisable. It is interesting to note that if we consider a variant of this polynomial, namely

$$(X - a_1)(X - a_2) \cdots (X - a_n) + 1, \tag{2}$$

then the polynomial is sometimes factorisable. For instance we have:

$$(X - 3)(X - 5) + 1 = (X - 4)^2.$$

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However, keeping $+1$ in place of -1, we can show that a large number of such polynomials are not factorisable. More precisely, we prove the following theorem.

Theorem 1. Let $n \geq 1$ be an integer and let $a_1, a_2, \ldots, a_n$ be distinct odd integers. Then the polynomial

$$f(X) = (X-2)(X-a_1)(X-a_2)\cdots(X-a_n)+1 \tag{3}$$

is not factorisable over the integers.

Remark 1. *The presence of the factor $(X-2)$ in Theorem 1 is crucial, otherwise the following serves as a counter-example:*

$$(X-3)(X-5)+1=(X-4)^2.$$

Remark 2. *As n is an arbitrary positive integer, the degree of f can be made arbitrarily large. Since the integers $a_1, \ldots, a_n$ vary over the odd natural numbers, these polynomials form an infinite family.*

Proof of Theorem 1. If possible, suppose that

$$f(X) = (X-2)(X-a_1)(X-a_2)\cdots(X-a_n)+1$$

is factorisable over the integers. Let $f(X) = g(X)h(X)$ for some two polynomials g and h with integer coefficients. We note that

$$f(2) = f(a_1) = \ldots = f(a_n) = 1.$$

Consequently, we have

$$g(2)h(2) = g(a_1)h(a_1) = \ldots = g(a_n)h(a_n) = 1. \tag{4}$$

Since g and h are polynomials with integer coefficients, $g(2), h(2), g(a_1), h(a_1), \ldots, g(a_n), h(a_n)$ are all integers. Therefore, from (4),

$$\left.\begin{aligned} &g(2) = h(2) = \pm 1,\\ &g(a_i) = h(a_i) = \pm 1 \text{ for all } i \in \{1, \ldots, n\}. \end{aligned}\right\} \tag{5}$$

Let $P(X) = g(X) - h(X)$. Then P is a polynomial with integer coefficients, and $P(2) = P(a_i) = 0$ for all $i \in \{1, \ldots, n\}$. Now,

$$\begin{aligned} \deg P(X) &= \deg(g(X) - h(X))\\ &\leq \max\{\deg(g(X)), \deg(h(X))\}\\ &< \deg(f(X)) = n+1. \end{aligned}$$

This means that $P(X)$ is a polynomial of degree $< n+1$, having at least $n+1$ distinct zeros (namely, $2, a_1, \ldots, a_n$). Hence $P(X)$ is identically zero, which implies that $g(X) = h(X)$.

Therefore, the relation $f(X) = g(X)h(X)$ becomes $f(X) = (g(X))^2$.

Let $g(0) = k$. Then from the equation $f(X) = (g(X))^2$, we get $f(0) = (g(0))^2 = k^2$. That is,

$$(-1)^{n+1} \cdot 2 \cdot a_1 a_2 \cdots a_n + 1 = k^2.$$

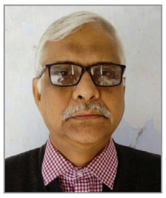
Since the a_i are all odd integers, k is odd. Hence $k^2 - 1 = (k-1)(k+1)$ is divisible by 8 (as it is the product of two consecutive even integers). This implies that $(-1)^{n+1} \cdot 2 \cdot a_1 a_2 \cdots a_n$ is divisible by 8, which is not possible as all the a_i are odd.

This contradiction shows that the stated factorisation is not possible.

This completes the proof of Theorem 1. $\square$

References

1. A. Engel, *Problem-Solving Strategies*, Springer.



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