Investigative Questions for the Middle School

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How is an investigation different from a problem? A problem generally has a unique solution that can be reached by applying standard procedures. In the case of an investigation, one is not sure at the outset if there is a solution at all or if there are multiple solutions. We may face novel situations requiring new approaches.

Here are two such questions. They require only middle school level arithmetic and algebra, but one has to search for possible solutions in a systematic way. The second one is much more demanding than the first.

I. Can a 3-digit number and the two 3-digit numbers obtained from it by cyclic permutation of its digits, form an arithmetic progression? That is, can the three 3-digit numbers *abc, bca* and *cab* form an A.P.? We should leave out trivial solutions such as 111, 222, etc.

   A variation on this theme would be to allow one of the digits to be zero. In that case one of the numbers in the set of three would actually be a 2-digit number, but one could consider that as having three digits, with zero in the hundreds place.

II. Can a 3-digit number and the two 3-digit numbers obtained from it by cyclic permutation of its digits, form a geometric progression? That is, can the three 3-digit numbers *abc, bca* and *cab* form a G.P.?

If you are not familiar with the term 'cyclic permutation,' here is an explanation. The triangle in Figure 1 has its vertices marked A, B, C. You may read the letters A, B, C in clockwise order, starting with each letter in turn. This gives the arrangements ACB, CBA and BAC. These three arrangements are cyclic permutations of each other. If the same is done moving in anti-clockwise direction, we obtain the arrangements ABC, BCA and CAB. These are again cyclic

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permutations of each other. Together they account for the six permutations of the three letters.

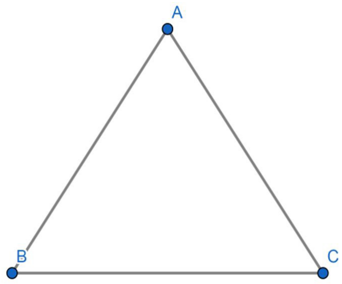


Figure 1

Solutions

b	a	c
1	1	1
2	2	2
3	3	3
4	4	4
	1	8
5	5	5
	2	9
	8	1
6	6	6
	9	2
7	7	7
8	8	8
9	9	9

Table 1

<u>I. If *abc, bca* and *cab* form an A.P.,</u> then
$2(100b + 10c + a) = 100a + 10b + c + 100c + 10a + b$, which simplifies to $7b = 4a + 3c$. We now need to find values of a, b, c from the set 1-9 that satisfy the above equation. We can assign values 1-9 to b in turn, and search for corresponding values of a and c. The possible values of a, b, c are given in Table 1.

We obtain the following solutions: (259,592,925), (148,481,814), (851,518,185), (962,629,296).

Allowing one of the digits to be zero, we have the following solutions: (037,370,703) and (740,407,074).

The common difference is ±333 in all cases.

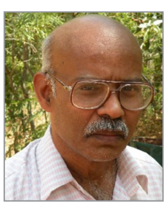
Now, why is this so? In all cases the common difference is $(100b + 10c + a) - (100a + 10b + c) = (-99a + 90b + 9c)$. We also know that a, b, c must satisfy the relation $7b = 4a + 3c$. Substituting for b from this relation, the expression for the common difference becomes, after simplification, $\frac{333}{7}(c - a)$. Since this quantity has to be an integer, $(c - a)$ must be a multiple of 7. Since both are single digit numbers, the value of $(c - a)$ can be only ±7; the only solutions for (c, a) are (7, 0), (0, 7), (8, 1), (1, 8), (9, 2) and (2, 9), which is reflected in our six solutions.

The reader is invited to find alternative approaches to solving the above problem.

<u>II. If *abc, bca* and *cab* form a G.P.,</u> then
$(100b + 10c + a)^2 = (100a + 10b + c)(100c + 10a + b)$, which, after multiplication, combining like terms, and cancelling common factors, reduces to $a^2 + 10ac = 10b^2 + bc$, or $10(b^2 - ac) = a^2 - bc$. As the RHS in the last equation is a multiple of 10, it suggests a way to check for possible solutions.

There are only two solutions: (432,324,243) and (864,648,486).

The numbers in the second set are all twice those in the first set. Naturally they have the same common ratio of 3/4.



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