Features

Spoof Numbers and Spoof Solutions - Part II

V G TIKEKAR

In this two-part article, we consider the curious notion of spoof numbers and spoof solutions which we get when we partially relax the conditions needed to define particular number families.

We recall, from Part I of the article [2], the definitions of *spoof number*, *spoof solution*, *Euler's sigma function* $S(n)$ and *perfect number*.

Definition 1 (Spoof number; spoof solution)**.** While constructing a number belonging to a particular family, if we relax some of the required properties or rules of formation but ensure that all the other properties of that family are satisfied, then such a number is called a *spoof number* of that family. Sometimes, such a number is also called a *quasi number* of that family. We similarly define the notion of *spoof solutions* by considering the spoof numbers obtained in the context of solutions of equations.

Definition 2 (Euler's sigma function)**.** If n is a positive integer, then $S(n)$ = sum of all the divisors of n.

Definition 3 (Perfect number)**.** A positive integer is called a *perfect number* if all its divisors add up to twice that integer, i.e., if $S(n) = 2n$.

The first few perfect numbers are 6, 28, 496 and 8128. We now explore the consequences of bringing these two notions together: spoof number and perfect number.

Keywords: Spoof number, spoof solution, Fermat number, perfect number, Mersenne number, triangular number, Euler

Odd perfect numbers

As of December 2018, 51 perfect numbers are known. Curiously, all of them are even numbers. To the great surprise of mathematicians and mathematics lovers, no one has been able to find any odd perfect numbers. At the same time, and to the still greater surprise of mathematicians and mathematics lovers, no one has been able to prove that there are no odd perfect numbers. All efforts in this direction, even by renowned mathematicians, have failed. Mathematicians have therefore reluctantly put forward the following:

Conjecture 1. *Odd perfect numbers do not exist.*

In passing, we note that the odd perfect number conjecture has been around for nearly 2000 years. The statement that all perfect numbers are even was first made around 100 CE by the Greek mathematician Nicomachus. So this is one of the oldest unsolved problems in mathematics.

Mathematicians have suggested that if odd perfect numbers are so difficult to find, then why not embark on looking for spoof odd perfect numbers? We shall do this later in the article.

Before that, we enumerate certain properties that odd perfect numbers must satisfy, if they exist. They have been proved over the centuries by various mathematicians.

Some properties of odd perfect numbers (OPNs), if they exist

(1) An OPN must have the form $p^{4a+1}m^2$ where p is a prime number of the type 1 (mod 4) and m is a natural number. This is Euler's characterization of an OPN. The prime p is called Euler's prime of the OPN.

(2) An OPN must have the form $12m + 1$ or $36m + 9$.

(3) An OPN must have at least 101 (not necessarily distinct) prime divisors.

(4) An OPN must have at least 7 distinct prime factors.

(5) The number of OPNs with k distinct prime factors is finite.

(6) An OPN with k distinct prime factors must be smaller than 2^{4^k}.

(7) No OPN can be divisible by 105.

(8) If an OPN is not divisible by 3, 5, 7, it must have at least 27 prime factors.

(9) The largest prime factor of an OPN must exceed 10^7.

(10) An OPN must be greater than 10^{2000}.

But no OPN has ever been found. Mathematicians have therefore looked at the properties that such numbers (if they do exist) must have, in the hope of finding contradictions between some of these properties (which would immediately show that these numbers do not exist). Unfortunately, all such efforts have failed. In fact, in 1888 Sylvester wrote:

> *The existence of an odd perfect number — its escape, so to say, from the complex web of conditions which hem it in on all sides — would be little short of a miracle.*

But nobody has been successful till now in proving the non-existence of an OPN. Since neither existence nor non-existence of OPNs is established, we consider OPNs to be hypothetical numbers and study them by relaxing some of the rules and try instead to obtain spoof OPNs. This thought leads us to the next section.

Spoof OPNs

In view of what has been noted above, let us relax our expectations and look for positive integers that behave like OPNs. In other words, let us look for spoof OPNs.

Descartes's spoof OPN. In 1638, René Descartes discovered the first spoof odd perfect number ("Descartes's number"):

$$D = 198,585,576,189.$$

To explain why it is a spoof OPN, we express the number in factorized form as follows:

$$D = 3^2 \times 7^2 \times 11^2 \times 13^2 \times 22021^1.$$

Among the numbers on the right, the only one which is not prime is $22021 = 19^2 \times 61$. But let us assume (incorrectly, of course) that 22021 is prime (this being our relaxation referred to above), and evaluate the S-function of this number. So we have (since 3, 7, 11, 13 and 22021 are coprime):

$$\begin{aligned}
&S(198,585,576,189)\\
&= S(3^2 \times 7^2 \times 11^2 \times 13^1 \times 22021^1)\\
&= S(3^2) \times S(7^2) \times S(11^2) \times S(13^2) \times S(22021^1) \quad \text{(by Rule 1; see [1])}\\
&= (1+3+3^2)(1+7+7^2)(1+11+11^2)(1+13+13^2)(1+22021) \quad \text{(by Rule 2)}\\
&= 397,171,152,378 = 2D.
\end{aligned}$$

We see that the Descartes number D satisfies the defining property of a perfect number, but as 22021 is not prime, D is not a true perfect number but rather a spoof perfect number. And since D is odd, D is a spoof OPN.

Comment. If we replace 22021 by its prime factors $19^2 \times 61$ and write D as

$$D' = 3^2 \times 7^2 \times 11^2 \times 13^1 \times 19^2 \times 61,$$

then D' will *not* be a spoof OPN.

Comment. Descartes believed that the number D would some day be modified to produce a genuine OPN. Let us hope that such a day will eventually dawn and mankind's efforts in the study of spoof numbers will be rewarded.

Voight's spoof OPN. This number was given by Voight more than three and a half centuries after Descartes, in 1999. The number is

$$V = -22,017,975,903.$$

To verify that this is an OPN:

$$\begin{aligned}
&S(-22,017,975,903)\\
&= S(3^4 \times 7^2 \times 11^2 \times 19^2 \times (-127)^1)\\
&= S(3^4) \times S(7^2) \times S(11^2) \times S(19^2) \times S([-127]^1) \quad \text{(by Rule 1)}\\
&= (1+3+3^2+3^3+3^4)(1+7+7^2)(1+11+11^2)(1+19+19^2)[1+(-127)] \quad \text{(by Rule 2)}\\
&= 2 \times (-22017975903).
\end{aligned}$$

Here the factor -127 is a negative integer and hence not prime (though 127 is prime), but we take it to be prime. So V is a spoof OPN.

Comment. Instead of V, if we consider the number $W = -V = 22,017,975,903$, then W will *not* be a spoof OPN.

Using computers to generate spoof OPNs. Note that the two spoof OPNs studied above (D and V) are big numbers, in contrast with the spoof even perfect numbers (like 60, 90, 84, 840) that we studied in Part I. Note also that it took several centuries to obtain the second spoof OPN (V) after the first OPN (D) came to light. Note further that the properties expected to be followed by OPNs are so involved that it would be extremely difficult to obtain OPNs by mere hand-calculation. All these considerations led mathematicians to use parallel computers (running for a few years non-stop) to look for spoof OPNs. (This mode of research has been tried out for other problems; for example, the four-colour problem, which was proved in 1976 after extensive use of computation.) This resulted in more spoof OPNs being found (including D and V).

A group of researchers ('BYU Computational Number Theory Group' – BYU being 'Brigham Young University') followed a systematic, computation-based research for obtaining more spoof OPNs. We give below three examples of the many spoof OPNs that this team discovered.

Example 1. We relax the usual conditions and assume that 1 is prime. So

$$S(1) = S(1^1) = (1+1) \quad \text{(by Rule 2)},$$

i.e., $S(1) = 2 = 2 \times 1 = 2$. Thus 1 is a spoof OPN.

Example 2. Consider the number $101,411,037$:

$$101,411,037 = 3^2 \times 7^2 \times 7^2 \times 13^1 \times (-19)^2.$$

Here we relax the rules we normally use by (i) writing 19^2 as $(-19)^2$ and treating -19 as prime; (ii) considering the two 7^2s as separate entities rather than considering them together as 7^4; and (iii) assuming that 7 and 7 are coprime so that 3, 7, 7, 13 and -19 are coprime (as is required to use Rule 1). So:

$$\begin{aligned}
&S(101,411,037)\\
&= S(3^2 \times 7^2 \times 7^2 \times 13^1 \times (-19)^2)\\
&= S(3^2) \times S(7^2) \times S(7^2) \times S(13^1) \times S[(-19)^2] \quad \text{(by Rule 1)}\\
&= (1+3+3^2)(1+7+7^2)(1+7+7^2)(1+13)[1+(-19)+(-19)^2] \quad \text{(by Rule 2)}\\
&= 202,822,074 = 2 \times (101,411,037).
\end{aligned}$$

So $101,411,037$ is a spoof OPN.

Comment. Readers can verify that if we take $(-19)^2$ as 19^2 (treating 19 correctly as prime), and/or take 7^4 in place of $(7^2) \cdot (7^2)$, then $101,411,037$ will no longer be a spoof OPN.

Example 3. Consider the number $11,025$:

$$11,025 = 1 \times 9 \times 25 \times 49 = (1^2) \times (-3)^2 \times (-5)^2 \times (49^1).$$

Here we take $9 = (-3)^2$ rather than 3^2; and similarly, $25 = (-5)^2$, and then wrongly assume that $1, -3, -5, 49$ are primes. Further, we take 1^2 as an additional factor, which does not change the value of the given number but we benefit by getting an increased value of the S-function. So:

$$\begin{aligned} &S(11025) = S[(1^2) \times (-3)^2 \times (-5)^2 \times (49^1)] \\ &= (1 + 1 + 1^2)[1 + (-3) + (-3)^2][1 + (-5) + (-5)^2](1 + 49) \quad \text{(by applying Rules 1 and 2)} \\ &= 3 \times 7 \times 21 \times 50 = 22050 = 2 \times 11025. \end{aligned}$$

Hence 11025 is a spoof OPN.

Remark. As mentioned above, spoof OPNs assume importance because we do not yet have any actual odd perfect numbers to exhibit. Mathematicians have adopted this kind of approach in the study of other classes of numbers whose existence is in doubt.

Spoof solutions related to Fermat's Last Theorem (FLT)

The notion of spoof solutions arises naturally when we are dealing with Fermat's Last Theorem (better known as FLT). We have all heard of Fermat's hand-written statement (1637): "It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers." In short:

$$\text{The equation } x^n + y^n = z^n \text{ has no solutions } (x, y, z, n) \text{ in positive integers for } n > 2. \tag{1}$$

In particular:

$$\text{The equation } x^{12} + y^{12} = z^{12} \text{ has no solutions } (x, y, z) \text{ in positive integers.} \tag{2}$$

Statement (1) has been proved in its most general form, so (2) is certainly true. That is, the equation (2) cannot be solved exactly over the positive integers. But in the context of our discussion about spoof numbers, the following question becomes meaningful:

Can we find spoof values of x, y, z so that the equation $x^{12} + y^{12} = z^{12}$ is satisfied with those values?

To proceed with the discussion, we must relax some of the normal rules. One way of proceeding is to move out of the set of positive integers, i.e., allow the use of non-integral numbers. This would mean that the above equation is satisfied up to a sufficient number of decimal places. For example, if we take

$$x = 3987, \quad y = 4365, \quad z = 4472,$$

then we find (using regular pocket hand-calculator) that

$$\left.\begin{array}{lclcl} x^{12} & = & 3987^{12} & \approx & 1.613447461 \times 10^{43}, \\ y^{12} & = & 4365^{12} & \approx & 4.784218174 \times 10^{43}, \\ z^{12} & = & 4472^{12} & \approx & 6.397665635 \times 10^{43}, \end{array}\right\} \tag{3}$$

then we find that the equation $x^{12} + y^{12} = z^{12}$ is satisfied up to 9 decimal places.

Concluding remarks

For many problems exact solutions either do not exist or are known not to exist. For example, (i) as of today, the existence or non-existence of OPNs is not known, and (ii) FLT is known to be true. By introducing the notion of spoof numbers and spoof solutions, we handle these situations and obtain inexact solutions, but they are solutions that imitate the behaviour of exact solutions, and so behave somewhat like them.

In the second part of this two-part essay, we have discussed what are known as spoof odd perfect numbers that behave somewhat like perfect numbers, provided we relax some of the usual rules. We also discussed what can be called spoof solutions to the equation that occurs in Fermat's Last Theorem. It is interesting that we can make some progress in these problems by allowing the relaxation of some conditions.

References

1. Jason Chuang, *Quanta Magazine*, "Mathematicians Open a New Front on an Ancient Number Problem", https://www.quantamagazine.org/mathematicians-open-a-new-front-on-an-ancient-number-problem-20200910/
2. V G Tikekar, *At Right Angles*, "Spoof numbers and spoof solutions - part I", http://publications.azimpremjifoundation.org/3047/
3. Wikipedia, "Descartes number" from https://en.wikipedia.org/wiki/Descartes_number
4. Wikipedia, "Divisor function" from https://en.wikipedia.org/wiki/Divisor_function

PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1995. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.