The Minimal Instruments of Geometry – II

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In the first part of this article, we introduced three alternative geometrical toolkits: (a) the straight edge and collapsible compass of Euclid, (b) the ruler, compass and protractor of Birkhoff and Beatley, and (c) the rope of the *Shulbasutras*. We also discussed some rope based geometrical constructions. In this second part, let us compare how these toolkits fare against some historical geometrical construction problems. We also ponder the construction of the 'tools' themselves, for example, how might we establish whether a straight edge is indeed straight and so on.

1. Toolkit Capabilities

Let us examine briefly, four constructions of interest during ancient times: doubling the cube, angle trisection, constructing a general regular polygon and squaring the circle [1]. It was eventually proven that none of these could be accomplished by Euclid's toolkit. As Birkhoff and Beatley summarized [2 pp. 165-166]:

' … [the construction of] a desired length *x*, from its relation to given lengths, *a, b, c,* etc., using only a straightedge and compasses… [is only possible] whenever the relationship of *x* to *a, b, c,* etc., involves ultimately only addition, subtraction, multiplication, division, and the extraction of square roots…'

'In general, it is impossible to divide a given angle into a given number of equal parts by means of straightedge and compasses alone.'

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In contrast, with the Birkhoff-Beatley toolkit, '… any construction involving the laying off of lengths and angles can be made with scale and protractor, and to any desired degree of accuracy' [2 p. 171]. The Birkhoff-Beatley toolkit can thus achieve all the four constructions under consideration.

Let us now study these four constructions in the context of the rope. It has been shown that the use of a markable straight edge and compass would make it possible to double a given cube and trisect a given angle [3, 4]. These constructions are thus possible with a rope. Angle trisection is also possible using the technique of finding rational multiples of a given angle with a rope as discussed in the first part of this article. This technique also enables the use of a rope to construct any regular polygon. As Figure 1 indicates, the angles that characterize a regular polygon are rational multiples of 360°.

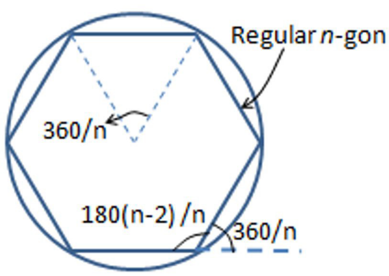

Figure 1. Various angles in a regular n-gon

The construction of a regular *n*-gon can proceed by constructing a side of the required length and then repeatedly making either the required interior angles $\left(= \frac{(n-2)}{2n} \times 360°\right)$, or the exterior angles (=(360°)/*n*) and marking off the required length on the new side thus formed. Another possibility is to construct the (360°)/*n* angles adjacently at the centre of a circle. The arms of these angles will then intersect the circle in points representing the corners of the required regular *n*-gon.

Finally, the rope can also be used to square the circle. Given a circle of radius *r*, our problem is to construct a square of area πr^2, i.e., a square with side $r\sqrt{\pi}$. We begin by marking off the lengths of the circumference and diameter on a couple of ropes. Having obtained these two lengths, we can divide the circumference by the diameter using straight edge and compass techniques, to get the value of π (Figure 2).

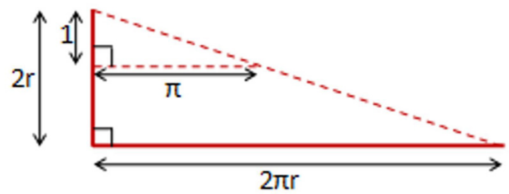

Figure 2. Calculating π from measured circumference and diameter of a circle

As mentioned earlier, given a length, we can find its square root, i.e., we can now get the length $\sqrt{\pi}$. Finally, we can multiply the two lengths *r* and $\sqrt{\pi}$ to construct the length $r\sqrt{\pi}$ (Figure 3) and subsequently a square with side of that length.

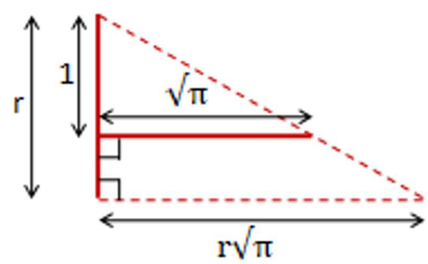

Figure 3. Multiplying the lengths r and √π

We note that though squaring a circle is thus possible using a rope, the *Shulbasutras* only contain approximate methods of squaring the circle [5, 6 pp. 143-149].

In theory, the Birkhoff-Beatley toolkit is still superior to the rope since it can construct a length and angle corresponding to any given number while the rope is limited to addition, subtraction, multiplication, division and square root constructions. However, given any number, we can always find a rational number as close to the given number as we want. Also, in practice, it would be impossible to mark the Birkhoff-Beatley tools with infinite resolution. This means that in practice, both the Euclid and the *Shulbasutra* toolkits should be able to yield geometrical constructions that are as accurate as the Birkhoff-Beatley toolkit constructions.

2. Tool Construction

Finally, we consider some of the practical challenges that inventors may have faced when first making the geometrical instruments discussed in this article.

2.1.1 Rope

The rope is arguably the easiest instrument to construct among those discussed here. In fact, ropes are naturally available in the form of vines, creepers, etc. In contrast, the toolkits of Euclid and Birkhoff-Beatley require a minimal degree of engineering expertise to precisely sculpt rigid objects into a desired shape. The finite thickness, limited flexibility and inconsistent elasticity of naturally occurring ropes pose challenges in accurate rope based constructions and make small scale geometrical constructions (e.g., which fit in a sheet of paper) almost impossible. The invention of cloth or thread making processes would resolve some of these difficulties by making possible ropes of near zero-thickness and high flexibility.

2.1.2 Compass

As we have seen, the rope itself can act as a compass. But even the contemporary compass shown in the first part of this article is fairly easy to make, for example, by using two sticks tied together at one end and the other end of each stick sharpened to a 'point'.

2.1.3 Straight Edge

How do we know whether a 'straight' edge is indeed straight? One way could be to line up a taut rope next to the edge to determine if it is straight. If it is not, it can be pared appropriately.

Another suggestion is to exploit the axiom: 'There is one and only one straight line between two given points' [2 p. 44]. In Figure 4, given two points P and Q, we use our 'straight' edge to draw a line joining them. We then turn our 'straight' edge around to swap its 'top' (T) and 'bottom' (B) ends and draw a 'straight' line again between P and Q. If the 'straight' edge is indeed straight, the two 'straight' lines thus drawn should exactly coincide.

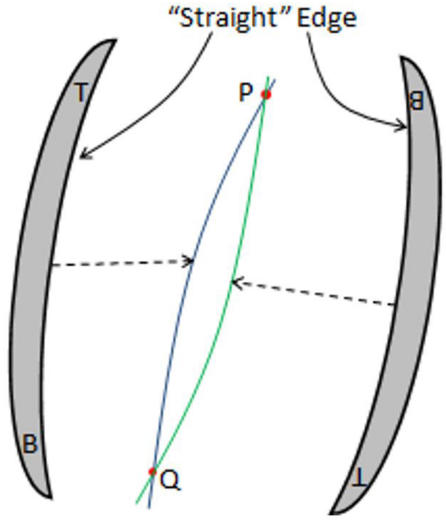

Figure 4. Test for straight edge

In Figure 4, the two lines do not coincide and we conclude that our 'straight' edge is in fact, not straight. We can also use the figure to tell where/how the 'straight' edge should be pared to make it straighter.

Does the ruler in your geometry box pass these tests of straightness?

2.1.4 Ruler

Given a straight edge, how does one mark it with infinite resolution? As mentioned earlier, this is impossible in practice. We can however, find a rational number as close as required to any given number and construct that length using the rope, or the straight edge and compass, which we now know how to make. We can use these constructions to mark the ruler.

One of the early small units of length was the width of a finger. Let us say this is standardised to 2 cm. Then, using the ruler of your geometry box as a straight edge along with the compass, how much further can you divide the 2cm length accurately? Can you construct a 1mm resolution ruler with these tools?

2.1.5 Protractor

The usual geometry box protractor is marked at every 1° angle from 0° to 180°. All these angles, being rational multiples of 360°, can be constructed with a rope.

Can we get a protractor marked at every 1° without using a rope? Euclid's *Elements* demonstrates the straight edge and compass construction of a regular 15–sided polygon. Further, angle bisection could be used to construct a regular $2n$-gon once a regular n-gon has been made. From the 15-sided polygon, we can thus arrive at 120, 240 and 480-sided regular polygons generating 3°, 1.5° and 0.75° angles (i.e., $(360°)/n$ angles) but not quite a 1° angle. In the 19th century it was proved that we cannot construct a 360-sided regular polygon using only a straight edge and compass [8]. However, using a markable straight edge, we can trisect the 3° angle to get a 1° angle.

Another option could be to use the 'table of chords' which had been developed to various degrees of accuracy by Hipparchus, Aryabhata, Ptolemy and possibly others [9, 10]. In modern terminology, this is equivalent to tables of the sine function. Ptolemy's table effectively listed the sine of angles from 0° to 90° in steps of 0.25°. In particular, we have sine(1°) ≈ 0.01745. We can use this to construct the triangle of Figure 5 and therefore a 1° angle. Note that this is still only an approximate construction since the value of sine(1°) can only be constructed to some rational approximation.

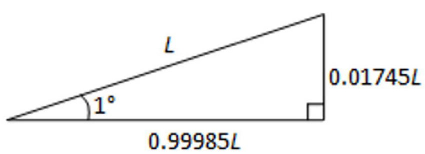

Figure 5. Constructing the required angle knowing its sine value

Ptolemy himself might have done this reverse construction using chords in a circle instead of right angled triangles in keeping with how he derived the sine tables.

Some practical difficulties remain with all the approaches discussed here. Can the length division of a rope be carried out to the extent required to generate a 1° angle with sufficient accuracy? Similarly, how accurate can the construction of a 3° angle with straight edge and compass be and further, how accurate can its trisection be? If we choose to use the table of chords and assume that a ruler with 1mm resolution is available to us, then in Figure 5, we will need to choose L~10m to create a 1mm difference in length between the base and the hypotenuse so that they can be accurately constructed to get a 1° angle between them. Is it easy to make a 10m ruler with better than 1mm accuracy and resolution?

3. Conclusion

In theory, the Birkhoff-Beatley toolkit can do many constructions that the Euclid or *Shulbasutra* toolkits can't. However, in practice, both the rope and the Euclidean toolkit should be able to achieve a similar accuracy to that of the Birkhoff-Beatley toolkit for any given construction. In fact, to 'invent' the Birkhoff-Beatley toolkit (i.e., to accurately mark the ruler and the protractor), we will need to use constructions based on the rope or the Euclid toolkit which in turn are constructible using intuitive (axiomatic) ideas. Can you think of a better way?

We also saw that the rope has some added advantages over both the Euclid and the Birkhoff-Beatley toolkits, due to its flexibility.

We reiterate that the ideas and techniques discussed in this article need not be a historically accurate account of how these developments actually took place. Rather, our attempt is to focus on ideas that are mathematically correct. In that spirit, we ask our readers what tools they would like in their toolkit to enable the construction of an increasing variety of curves and shapes? How can these tools themselves be constructed?

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