Student Corner

A Geometry Problem from IMO 2016

SHREYAS ADIGA

In this article, we solve a geometry problem from the International Mathematics Olympiad (IMO) 2016 (Hong Kong). It was proposed by Art Waeterschoot from Belgium, who received an honourable mention during the IMO 2015 (Thailand). The problem was given to the problem-solving group of our school.

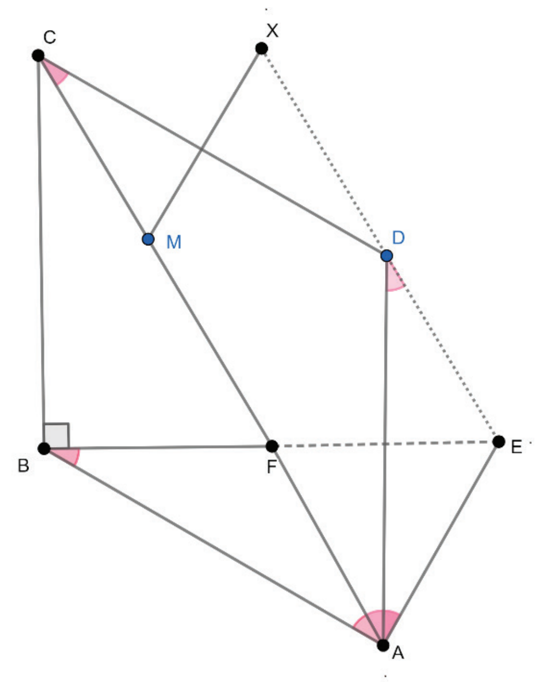

Figure 1

Keywords: Triangle, parallelogram, concurrent, radical axis, concyclic, IMO, INMO, USAMO

Problem 1. Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen such that $DA = DC$ and AC is the bisector of $\measuredangle DAB$. Point E is chosen such that $EA = ED$ and AD is the bisector of $\measuredangle EAC$.

Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$).

Prove that lines BD, FX and ME are concurrent.

Solution

We find three circles such that BD, FX, ME are the respective radical axes associated with pairs of these circles. Then we apply the radical axis theorem.

From the data of the problem, we can denote $\measuredangle CAB = \measuredangle FAD = \measuredangle DAE = \measuredangle EDA = \theta$.

The main claims in the proof are the following:

- Points D, F, B, C, X are concyclic.
- Points B, A, E, D, M are concyclic.
- Points E, F, M, X are concyclic.
    - **Claim 1**: Points E, D, X are collinear.

      Proof: Since AD is the angle bisector of $\measuredangle EAC$, we have $\measuredangle CAD = \measuredangle EAD$.

      Since $ED = AE$, $\triangle AED$ is isosceles and hence $\measuredangle EAD = \measuredangle EDA$. Thus $\measuredangle CAD = \measuredangle EDA$.

      This implies that $ED \parallel AC$.

      Now, as $AMXE$ is a parallelogram, $EX \parallel AC$. It follows that E, D, X are collinear (Figure 1).

    - **Claim 2**: Points D, F, B, C are concyclic.

      Proof: From the observation that $\triangle ABF$ is similar to $\triangle ACD$, we have,

      $$\frac{AB}{AC} = \frac{AF}{AD}.$$

      Next, observe that $\measuredangle BAF = \measuredangle FAD$. It therefore follows that $\triangle ABC$ is similar to $\triangle AFD$.

      We further note that $\measuredangle AFD = \measuredangle ABC = 90^\circ + \theta$. Since $\measuredangle DCF = \theta$ and $\measuredangle AFD$ is the exterior angle of $\triangle CFD$, we conclude that $\measuredangle FDC = 90^\circ$ and hence B, C, D, F are concyclic with CF as diameter. Denote this circle by Γ_1 (Figure 2).

    - **Claim 3**: B, E, F are collinear.

      Proof: Observe that $\triangle CDA$ is isosceles, with $\measuredangle DCA = \measuredangle DAC = \theta$.

      From the previous claim, we know that B, C, D, F are concyclic. Hence $\measuredangle FBD = \measuredangle FCD = \theta$ (as these are angles on the same segment FD).

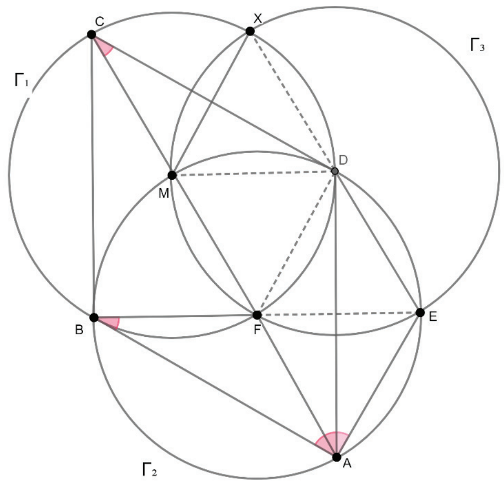

Figure 2

From the observation that $\triangle ABF$ is similar to $\triangle ADE$ we have:

$$\frac{AB}{AD} = \frac{AF}{AE}.$$

Since $\measuredangle BAD = \measuredangle FAE$, it follows that $\triangle ABD$ is similar to $\triangle AFE$.

Now $\measuredangle AFE = \measuredangle ABD = \theta + \measuredangle FBD = \theta + \measuredangle FCD = \theta + \theta = 2\theta$.

But in $\triangle ABF$, $\measuredangle AFE = 2\theta = 180 - \measuredangle BFA$. This proves the claim (Figure 1).

- **Claim 4**: A, B, M, D are concyclic.

  Proof: Since B, C, D, F are concyclic (from Claim 2) with M as centre of the circumcircle of the $\triangle FBC$, $\measuredangle AMD = \measuredangle FMD = 2\measuredangle ACD = 2\theta$.

  Again, B, C, D, F being concyclic implies that $\measuredangle EBD = \theta$, and hence $ABD = \measuredangle ABE + \measuredangle EBD = \theta + \theta = 2\theta$.

  We conclude that A, B, M, D are concyclic. Denote this circle by $\mathbf{\Gamma}_2$ (Figure 2).

- **Claim 5**: E lies on $\mathbf{\Gamma}_2$.

  Proof: From Claim 2, we know that B, C, D, F are concyclic, so $\measuredangle FBD = \measuredangle FCD = \theta$.

  Since E, F, B are collinear, $\measuredangle EBD = \measuredangle FBD = \theta$.

  But $\measuredangle EAD = \theta$. Thus E, A, B, D are concyclic.

  Now, from Claim 4, A, B, M, D are concyclic and hence E lies on $\mathbf{\Gamma}_2$.

- **Claim 6**: X lies on $\mathbf{\Gamma}_1$.

  Proof: From the observation that $MDEA$ is cyclic, it follows that $\measuredangle MDX = \measuredangle EAM$. Since $AMXE$ is a parallelogram, $\measuredangle EAM = \measuredangle DXM$.

  Therefore, $\triangle MDX$ is isosceles and so $MD = MX$, thus proving the claim (Figure 2).

- **Claim** 7: M, F, E, X are concyclic.

  Proof: We observe that $\measuredangle AFE = 2\theta$ (this is the exterior angle for the triangle AFB). So $EF = EA$.

  Since $AEXM$ is a parallelogram, $EA = MX$. Therefore $EF = MX$ and so $MFEX$ is an isosceles trapezium. It follows that M, F, E, X are concyclic (an isosceles trapezium is always cyclic). Let us denote circle $MFEX$ by Γ_3.

  Now observe that BD, FX, ME are the radical axes of the circle pairs (Γ_1, Γ_2), (Γ_3, Γ_1), (Γ_2, Γ_3). By applying the radical axis theorem, we get the desired result. (The Radical Axis Theorem states: *The three pairwise radical axes of three circles concur at a point*. The point where the lines meet is called the 'radical centre' of the three circles.)

Two problems for the reader

Problem 2 (USAMO 1997)**.** Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

(Here, USAMO refers to the Mathematics Olympiad held in USA.)

Problem 3 (IMO 1995)**.** Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. Line XY meets BC at Z. Let P be a point on line XY other than Z. Line CP intersects the circle with diameter AC at C and M, and line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

References

Evan Chen, *Euclidean Geometry in Mathematical Olympiads*, Mathematical Association of America (MAA), 2016.

SHREYAS S ADIGA is a student of class 12 at the Learning Centre PU College, Mangalore, Karnataka. He has a deep love for problem solving in Mathematics, especially Geometry and Number theory. He received a Winner diploma in 16th I F Sharygin Geometry Olympiad. He also got HM in APMO 2022. He cleared the Indian National Mathematics Olympiad (INMO) in 2021 and 2022. He has a keen interest in chess. He may be contacted at ssa135228@gmail.com.