Student Corner

Solution to a 2018 Romanian Mathematics Olympiad Problem

PRANEETHA KALBHAVI

In this article, we solve an inequality problem involving logarithms.

Problem (Romania MO 2018)**.** Let a, b, c be real numbers such that $1 < b \leq c^2 \leq a^{10}$ and

$$\log_a b + 2\log_b c + 5\log_c a = 12.$$

Show that

$$2\log_a c + 5\log_c b + 10\log_b a \geq 21.$$

Solution. Let $\log_c b = k$, $\log_a c = l$, $\log_a b = m$. From $1 < b \leq c^2 \leq a^{10}$, we have

$$\log_c b \leq 2, \quad \log_a b \leq 10, \quad \log_a c \leq 5, \tag{1}$$

i.e., $0 < k \leq 2,\ 0 < m \leq 10,\ 0 < l \leq 5$. The hypothesis is now transformed to

$$m + \frac{2}{k} + \frac{5}{l} = 12. \tag{2}$$

Noting that $kl = m$, (2) gets simplified to

$$2l + 5k = m(12 - m). \tag{3}$$

The goal now becomes: show that

$$2l + 5k + \frac{10}{m} \geq 21. \tag{4}$$

Keywords: Logarithms, Romanian Mathematics Olympiad

In view of (3), the goal (4) further simplifies to:

$$m(12 - m) + \frac{10}{m} \geq 21,$$

which in turn is equivalent to

$$m^3 - 12m^2 + 21m - 10 \leq 0. \tag{5}$$

To see why (5) is true, we observe that $m - 1$ is a factor of the polynomial on the left side, for $1 - 12 + 21 - 10 = 0$. After performing the relevant division, we get

$$m^3 - 12m^2 + 21m - 10 = (m - 1)(m^2 - 11m + 10),$$

and we see that $m - 1$ is a factor of $m^2 - 11m + 10$ as well, for $1 - 11 + 10 = 0$. We have:

$$m^2 - 11m + 10 = (m - 1)(m - 10).$$

Hence

$$\begin{aligned} & m^3 - 12m^2 + 21m - 10 \\ & = (m - 10)(m - 1)^2 \\ & \leq 0, \quad \text{since } 0 < m \leq 10. \end{aligned}$$

The proof is now complete. □

References

1. *Crux Mathematicorum* , 2020, No. 46 (6)

PRANEETHA KALBHAVI is currently a Class 12 student at The Learning Center PU College, Mangalore, Karnataka. She has deep love for Mathematics and Science. She participated in INMO 2020. Her hobbies include reading and badminton. She wishes to pursue a career in mathematics or computer science. She may be contacted at praneethakalbhavi@gmail.com.