Finding the Base Angles of a Triangle

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Consider a $\triangle ABC$ in which the following are specified: $\measuredangle A$ (i.e., the apex angle BAC), the length a of the base BC, and the length h of the altitude from A to BC.

Is it possible to find expressions for the two base angles, $\measuredangle B$ and $\measuredangle C$, in terms of A, a, h? We do so using trigonometry.

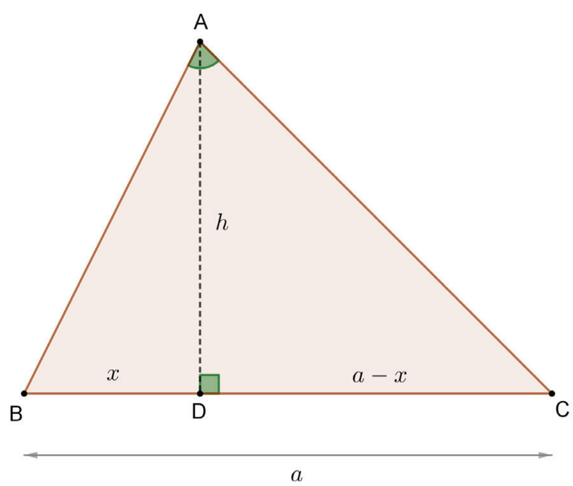

Figure 1.

Let AD be the perpendicular from vertex A to BC, and let $BD = x$, $DC = a - x$. (For convenience, we assume that $\measuredangle B$ and $\measuredangle C$ are acute, which means that D lies on the side and not on the extension of the side. We also assume that the triangle is not right-angled.)

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From right-angled triangles ABD and ACD, we have:

$$\tan B = \frac{h}{x}, \quad \tan C = \frac{h}{a-x}. \tag{1}$$

Hence:

$$x = \frac{h}{\tan B}, \quad a - x = \frac{h}{\tan C} = -\frac{h}{\tan(A+B)}, \tag{2}$$

where the last step comes from the fact that $C = 180° - (A+B)$. Hence:

$$\begin{aligned}\frac{a}{h} &= \frac{\tan(A+B) - \tan B}{\tan B \cdot \tan(A+B)}\\ &= \left(\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} - \tan B\right) \div \tan B \cdot \left(\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}\right)\\ &= \frac{\tan A + \tan A \cdot \tan^2 B}{\tan B \cdot (\tan A + \tan B)}.\end{aligned}$$

From the last relation we obtain, by cross-multiplication:

$$(a - h\tan A)\tan^2 B + (a\tan A)\tan B - h\tan A = 0. \tag{3}$$

Here, (3) can be regarded as a quadratic equation in $\tan B$; the coefficients are known quantities, as they have been expressed in terms of a, h, A. Solving the equation, we get:

$$\tan B = \frac{-a\tan A \pm \sqrt{a^2\tan^2 A + 4(a - h\tan A)\cdot h\tan A}}{2(a - h\tan A)}. \tag{4}$$

We have not attempted to simplify the expression in (4). An equivalent way of expressing the same result, in terms of sines and cosines, is the following:

$$\tan B = \frac{-a\sin A \pm \sqrt{a^2\sin^2 A + 4(a\cos A - h\sin A)\cdot h\sin A}}{2(a\cos A - h\sin A)}. \tag{5}$$

Note the plus-minus sign. The two values given by the formula correspond to the values of $\tan B$ and $\tan C$ respectively. (There is an obvious symmetry in the problem between B and C.)

If the triangle is right-angled, then we may encounter fractions with zero denominator, so we need to be careful. We look at this possibility below.

The case when A $= 90°$**.** In this case, $\measuredangle B + \measuredangle C = 90°$, so $\tan B \cdot \tan C = 1$. Therefore (2) assumes the form

$$x = \frac{h}{\tan B}, \quad a - x = \frac{h}{\tan C} = h\tan B, \quad \therefore\ x(a-x) = h^2. \tag{6}$$

The quadratic equation $x(a-x) = h^2$ may be solved for x, and from this we get $\tan B$:

$$x(a-x) = h^2, \quad \therefore\ x = \frac{a \pm \sqrt{a^2 - 4h^2}}{2},$$

$$\therefore\ \tan B = \frac{2h}{a \pm \sqrt{a^2 - 4h^2}}. \tag{7}$$

Rationalising, we get:

$$\tan B = \frac{a \mp \sqrt{a^2 - 4h^2}}{2h}. \tag{8}$$

As earlier, the two values given by the formula correspond to the values of $\tan B$ and $\tan C$ respectively. (Note that the product of the two values is equal to 1, as it should be.)

The case when a denominator of 0 occurs in (4) **and** (5)**.** This will happen when $a = h \tan A$ (equivalently, $a \cos A = h \sin A$). This means, clearly, that either $\measuredangle B = 90^\circ$ or $\measuredangle C = 90^\circ$.

References

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