Problem Corner

An Arithmetic Escapade

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Problem. Consider the numbers 75, 84 and 93. They are in arithmetic progression. Flip the digits to obtain 57, 48 and 39. These three numbers too are in arithmetic progression. Can we characterise all such three-term arithmetic progressions consisting of two-digit positive integers?

Solution. Let the three numbers be

$$10a_1 + b_1, \quad 10a_2 + b_2, \quad 10a_3 + b_3$$

where $a_i, b_i \in \{1, \ldots, 9\}$ for $i = 1, 2, 3$. Note that we have assumed that the digits are non-zero so that the case of having a single digit number upon flipping is eliminated. Since

$$10a_1 + b_1, \quad 10a_2 + b_2, \quad 10a_3 + b_3$$

are in arithmetic progression we have

$$10(a_1 - 2a_2 + a_3) + (b_1 - 2b_2 + b_3) = 0.$$

Also, the numbers $10b_1 + a_1$, $10b_2 + a_2$, $10b_3 + a_3$ are in arithmetic progression, which implies

$$10(b_1 - 2b_2 + b_3) + (a_1 - 2a_2 + a_3) = 0.$$

Therefore

$$a_1 - 2a_2 + a_3 = b_1 - 2b_2 + b_3 = 0.$$

Thus the corresponding digits of the given positive integers are in arithmetic progression.

Keywords: Arithmetic progression, digits

The case of three-digit numbers. What if we want to characterise three-digit positive integers with the same property? Let the numbers

$$100a_i + 10b_i + c_i$$

with $a_i, b_i, c_i \in \{1, \ldots, 9\}$, $i = 1, 2, 3$ be such that the numbers

$$100a_1 + 10b_1 + c_1, \quad 100a_2 + 10b_2 + c_2, \quad 100a_3 + 10b_3 + c_3$$

and

$$100c_1 + 10b_1 + a_1, \quad 100c_2 + 10b_2 + a_2, \quad 100c_3 + 10b_3 + a_3$$

form two arithmetic progressions. Then

$$100(a_1 - 2a_2 + a_3) + 10(b_1 - 2b_2 + b_3) + (c_1 - 2c_2 + c_3) = 0$$

and

$$100(c_1 - 2c_2 + c_3) + 10(b_1 - 2b_2 + b_3) + (a_1 - 2a_2 + a_3) = 0.$$

Whence, by subtraction,

$$a_1 - 2a_2 + a_3 = c_1 - 2c_2 + c_3 = t(\text{say}).$$

Then

$$10(b_1 - 2b_2 + b_3) = -101t.$$

Observe that $10|t$ and $|t|< 20$. Thus $t \in \{0, \pm 10\}$. If $t = \pm 10$ then $b_1 - 2b_2 + b_3 = \mp 101$, which is absurd. Hence $t = 0$ and $b_1 - 2b_2 + b_3 = 0$. We see that for three-digit positive integers to possess the desired property, the corresponding digits must be in arithmetic progression.

The general case. Now we proceed to tackle the case of positive integers with n (> 3) digits. Let the numbers

$$x_i = \sum_{k=1}^{n} a_{i,k} 10^{k-1}$$

with $a_{i,k} \in \{1, \ldots, 9\}$, $i = 1, 2, 3$ and $k = 1, \ldots, n$ be such that x_1, x_2, x_3 is an arithmetic progression and so are the numbers y_1, y_2, y_3 where

$$y_i = \sum_{k=1}^{n} a_{i,n+1-k} 10^{k-1}.$$

By setting $\lambda_k = a_{1,k} - 2a_{2,k} + a_{3,k}$, it follows that

$$\sum_{k=1}^{n} \lambda_k 10^{k-1} = 0 \qquad (1)$$

and

$$\sum_{k=1}^{n} \lambda_{n-k+1} 10^{k-1} = 0. \tag{2}$$

Observe that $10|\lambda_1$ and $10|\lambda_n$. Since $|\lambda_1| < 20$ and $|\lambda_n| < 20$, we must have $\lambda_1, \lambda_n \in \{-10, 0, 10\}$. Suppose $\lambda_1 \neq 0$. Then by (2) we have

$$\lambda_1 10^{n-1} = -\sum_{k=1}^{n-1} \lambda_{n-k+1} 10^{k-1}.$$

But since $|\lambda_{n-k+1}| < 20$ for $1 \leq k \leq n$ we have

$$\left| -\sum_{k=1}^{n-1} \lambda_{n-k+1} 10^{k-1} \right| < \frac{2(10^n - 10)}{9} < 10^n = |\lambda_1|\, 10^{n-1},$$

a contradiction. Hence $\lambda_1 = 0$.

Proceeding in a similar manner by using equation (1) we can show that $\lambda_n = 0$. Thus the terms independent of a power of 10 in both equations (1) and (2) are zero.

On dividing both sides of (1) and (2) by 10 we obtain two equations:

$$\sum_{k=1}^{n-2} \lambda_{k+1} 10^{k-1} = 0 \tag{3}$$

and

$$\sum_{k=1}^{n-2} \lambda_{n-k} 10^{k-1} = 0. \tag{4}$$

The same argument may be repeated with equations (3) and (4) to show that the constant terms are zero and the process continues till we show that $\lambda_k = 0$ for $1 \leq k \leq n$. This implies that the corresponding digits of the given numbers must form an arithmetic progression.

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