TearOut

Polyominoes and nets of a cube

In this 6th TearOut, we will explore polyominoes and all possible nets of cubes (and cuboids). As before, pages 1 and 2 are a worksheet for students while pages 3 and 4 give guidelines to the facilitator. Though dot sheets or square grid sheets are not a must, they may be useful to make the set of all polyominoes (up to hexominoes). Using these will help with the exploration.

ClassRoom

Each polyomino is made of one or more congruent squares. Some of the sides of these squares are inside the polyomino while the rest are outside, forming the perimeter of the same. We are going to call each of the square-sides along the perimeter of the polyomino as an 'out-line'.

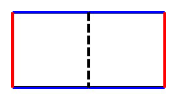

Figure 1

Now, observe that the monomino, which is just a square, has only one type of out-line because a monomino has rotational symmetry of order 4. So, if ABCD is the monomino, the out-line AB can coincide with BC, CD or AD if we chose a suitable rotation (or a reflection). Therefore, it doesn't matter where we add the next square to get a domino. So, there is only one type of domino, modulo rotation. However, there are two types of out-lines for the domino – (i) the red out-lines, which can be reflected on each other along the dashed line in the middle as shown in Figure 1, and (ii) the 4 blue out-lines which coincide with each other if we rotate by 180° or reflect along the perpendicular bisector of the red out-lines. So, we get two types of trominoes depending on where we attach the third square (Figure 2).

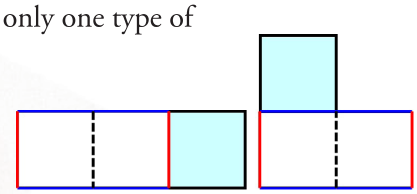

Figure 2

Also, if a polyomino is obtained by adding a square to an existing polyomino then the former is called a 'child' of the latter and the latter is a 'parent' of the former. So, monomino is the parent of domino and trominoes are children of the latter.

1. Types of 'out-lines' and number of children
    a. Find out how many types of out-lines there are for each tromino. So, how many possible tetrominoes are there? [Hint: consider the symmetries of each tromino]
    b. Is there any tetromino with two parents? If so, which one(s)?

2. Parents
    a. Now consider the 12 possible pentominoes (Figure 3). Find the parent(s) for each.
    b. Which pentominoes have only one parent? Which ones have two? Is there any with more than two parents? Create a suitable chart.
    c. Complete the tree diagram (Figure 4) to include all tetrominoes and all pentominoes.

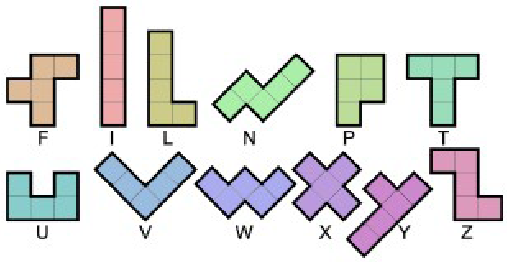

Figure 3: By R. A. Nonenmacher - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=4412149

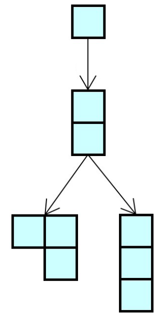
Figure 4

3. Wrapping around the cube

Monomino coincides with any face of a cube (of the same size). Similarly, domino can be folded and placed against a cube such that the two squares superimpose on two adjacent faces of the solid. So, when a polyomino is wrapped around a cube, each square must superimpose on a unique face of the solid. Therefore, the squares of a polyomino should not overlap with each other, neither should any square be left out, i.e., not overlapping a cube-face.

a. Can both trominoes be wrapped around the cube?
b. Is there any tetromino that can't be wrapped around the cube? Guess and reason.
c. Based on the above, which pentomino(es) cannot be wrapped around the cube?
d. Which other pentominoes cannot be wrapped around the cube? Guess and reason.
e. When a pentomino is folded to form a cube, how should it look?
f. List all pentominoes that can be wrapped around a cube.

4. Net of a cube

A cube has 6 square faces. So, if a cube is opened up to form its net, it is made of 6 squares, i.e., a hexomino. These are all the possible hexominoes (Figure 5).

a. If a polyomino cannot be wrapped around a cube, can its children be? Why?
b. Use above and 3f (cleverly) to strike out the hexominoes that cannot be the net of a cube. Give reasons for each.
c. A cube's net can have 4 squares in a line (the walls) forming a 4×1 rectangle, with the remaining squares (floor and roof) on each of the longer sides of the rectangle. Identify the hexominoes fitting this description.
d. Fold each of the remaining hexominoes and find out which ones are nets of a cube.
e. How many possible nets are there?

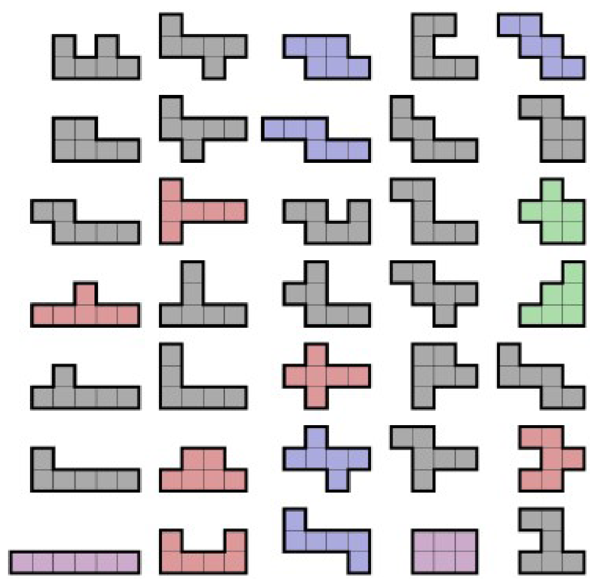
Figure 5: By R. A. Nonenmacher - Created by me, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=4773113

5. [Optional] Net of a cuboid

a. Suppose the cube has been flattened to form a cuboid with dimensions $a \times a \times b$ with $0 < b < a$. How will each of the nets of the cube change?
b. Repeat the above for an elongated cube, i.e., a cuboid with dimensions $a \times a \times b$ with $0 < a < b$.
c. Repeat the above for a cuboid with dimensions $a \times b \times c$ with $0 < a < b < c$.

This worksheet touches upon many ignored areas of mathematics. It starts with communication including creating terminology to convey ideas precisely and rigorously as per the need. Therefore, we introduce 'out-lines' as well as the children-parent relation among the polyominoes. This way the tree-diagram of polyominoes can be linked to the family tree – something that students may already be familiar with. It also fosters reasoning at every step.

1. This is a warm-up activity to get familiar with types of 'out-lines' for each polyomino and the notion of children and parent of the same. It is best not to show all possible tetrominoes to the students but let them discover the same. One may note that each child is a superset of a parent.
2. While the first question involved finding all children of the trominoes, the second one reverses the process and is about finding the parents. Since each parent is a subset of a child, one has to remove a square from the child to get a parent.

   The task ends with extending the 'family tree' of the polyominoes till the 5th generation, i.e., the pentominoes. This tree is different since many branches often merge – something impossible in a usual family tree.
3. In this question, we go to the second part – nets of a cube. One can imagine wrapping trominoes around a cube. But it would help to actually have paper polyominoes and fold them to form partial cubes.

   While both trominoes do wrap around a cube, there is one tetromino that doesn't. Let us call this tetromino O. Only three squares meet at any vertex of a cube. So, if a polyomino has four squares sharing a vertex then it can't be wrapped around a cube. Consequently, the pentomino P which is the only child (why only?) of tetromino O cannot be wrapped around a cube.

   I is the other pentomino that can't be wrapped around a cube. This is because there are only four squares as we go around a cube forming a loop. So, the 5th square of I becomes extra. The pentominoes V and U also fall in this category. However, it may be difficult to reason out why that is the case. But if one tries to fold a V or a U to a cube, it becomes clear.

   Each pentomino has five squares. So, if one can be wrapped around a cube, then the folded one will look like a hollow cube with one face missing.
4. In this part we narrow down the list of all hexominoes to the possible nets of a cube. This is done in three ways – (i) elimination, (ii) justifying why it can be a net, and (iii) by actually folding to form a cube.

   We get a child by adding a square to the parent. So, if the parent can't be wrapped around a cube, adding one more square won't change that. However, note that the converse is not true since the parent of tetromino O can be wrapped around a cube.

   Using the above argument we can eliminate 21 out of the 35 hexominoes. These 21 are children of pentominoes P, I, V or U – the 'or' is very inclusive here!

   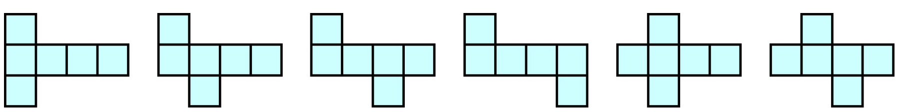

   Figure 6

   Six of the remaining hexominoes can be justified by the 4-walls-roof-floor argument (Figure 6). Four more can be considered as variations of some of these six (Figure 7).

   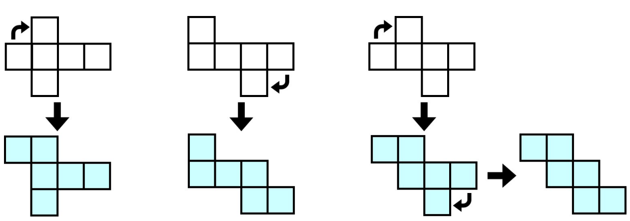

   Figure 7

That leaves just four hexominoes which can be checked by folding. One of them does form a cube (Which one?). So, there are 11 possible nets of a cube.

5. This optional task stretches the exploration to cuboids by changing one dimension at a time. This can be given to groups of students, each exploring some of the 11 nets. Once all 11 nets are obtained for each set, there can be a further exploration on optimization using algebra.

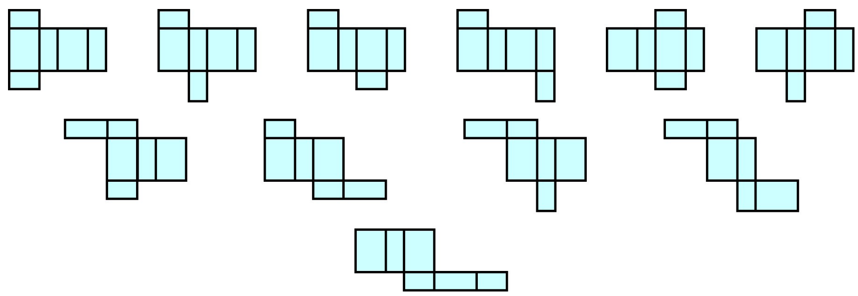

Figure 8: all possible nets for cuboid $a \times b \times c$ with $0 < a < b < c$

Suppose each of these nets has to be cut out of a rectangular piece of cardboard.

a. What would be the dimensions of the smallest rectangle for each?

b. How many pieces would have to be cut out of the rectangle to get the net? Note that fewer number of pieces to cut out means more efficiency with respect to effort.

c. What is the total area of the pieces cut out (and thrown away)? Again, the smaller the area, the better optimization of the material, i.e., cardboard.

d. Use the above to find the optimum net for each set, i.e., (i) $a \times a \times b$ with $0 < b < a$, (ii) $a \times a \times b$ with $0 < a < b$, and (iii) $a \times b \times c$ with $0 < a < b < c$.

e. Can you arrange all 11 nets of each of the three sets from most to least economical? Are there some at the same optimum level? Are there any that are difficult to fit in the continuum? Why?

Questions d and e are best explored with some random numbers first and then algebraically. Set (i) generates 3 sizes of rectangles – $(2a + b)(4a)$, $(2a + b)(3a + b)$ and $(a + b)(4a + b)$, which can be easily ordered since $0 < b < a$. Similarly, set (ii) generates the same three sizes algebraically speaking. But because $0 < a < b$, the ordering can only be partial. Set (iii) however generates 5 rectangles: $(2a + c)(2a + 2b)$, $(a + b + c)(2a + 2b)$, $(2a + c)(a + 2b + c)$, $(a + b + c)(a + 2b + 2c)$ and $(a + c)(a + 2b + 2c)$. Some partial ordering can be argued among these, based on $0 < a < b < c$. These algebraic arguments would require playing with inequalities but that shouldn't be too hard since we are considering positive quantities only. It is worth pointing out that some of the most economical nets are usually seen while the less economical ones are rarely found anywhere.

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