Responses to: Did Calculus Change the World?

I. The Survey

(i) Could you briefly describe for a layperson the problem(s) you work on?

(ii) Can you explain how calculus plays a part in your work (again as non-technically as possible)?

(iii) Steven Strogatz begins his book, Infinite Powers, with the following claim (page vi of the Introduction)

> "Without calculus, we wouldn't have cell phones, computers, or microwave ovens. We wouldn't have radio. Or television. Or ultrasound for expectant mothers, or GPS for lost travellers. We wouldn't have split the atom, unravelled the human genome, or put astronauts on the moon. We might not even have had the Declaration of Independence.
>
> It's a curiosity of history that the world was changed forever by an arcane branch of mathematics."

Later on he tempers the above statements by saying (page x of the introduction):

> "...This is a much broader view of calculus than usual. It encompasses the many cousins and spinoffs of calculus, both within mathematics and in adjacent disciplines. Since this big-tent view is unconventional, I want to make sure it doesn't cause any confusion. For example, when I said earlier that without calculus we wouldn't have computers and cell phones and so on, I certainly didn't mean to suggest that calculus produced all these wonders by itself. Far from it. Science and Technology were essential partners-and arguably the stars of the show. My point is merely that calculus has also played a crucial role, albeit often a supporting one, in giving us the world we know today."

(on page xi after describing Maxwell's work on electromagnetism)

> "Clearly, calculus could not have done this alone. But equally clearly, none of it would have happened without calculus. Or, perhaps more accurately, it might have happened, but only much later, if at all."

What is your opinion about this?

II. Professor Ravi Prakash Jagadeeshan

The problems I work on: I am interested in a field known as "Polymer dynamics and rheology."

Polymers are very large molecules, composed of many smaller molecules called monomers. They are usually very long, like a piece of string or spaghetti, and in this case they are called linear polymers. However, they can also have complex architectures such as stars, or rings with no ends, or have branched topologies like combs and even have hierarchically branched structures like dendrimers, all made up of many monomers attached together.

I am interested in the properties of solutions of polymers, and the way they behave when they flow.

Knowing this is important for various reasons. All the plastics around us are processed with polymers either in solution or in the melt state. We may want to know how they behave when they go through various types of machinery, such as pumps or extruders. What is the rate at which they will flow, how much force will they exert on the walls of the geometry they are flowing in. How much force should we apply to push the solutions through such devices, etc. Importantly, most molecules in biology are polymers as well, such as DNA, proteins, etc. Many physical properties of biological systems can be understood by studying them from the perspective of polymer physics. For instance how does DNA fold itself into the very small dimensions of a cell's nucleus. How does a protein fold into its final state after it is made. The 3D structure of a protein is extremely important for its function.

The aim of my research is to develop molecular theories using simple models for polymers, such as beads connected to each other with springs, and to use these theories to predict the behaviour of solutions. Such simple models account for the fact that a polymer is a long chain molecule made up of many monomers (or beads), that a polymer experiences the drag when it moves through a solution, and that it can be stretched and oriented.

Many properties of polymer solutions only depend on these aspects of polymers; the details at the level of chemistry don't affect the qualitative behaviour when the properties involve large length and time scales of the order of the size of the polymer and not of the monomers of which they are made. For instance, the viscosity of a polymer solution, which describes how easily the solution will flow if one deforms it, is such a macroscopic property.

Once we have a successful molecular theory we can run computer simulations to predict how a system composed of polymers will behave, and we can study many of the problems I gave earlier as examples.

On the role of calculus: Once we have developed a model for a polymer, we want to know how an ensemble of polymers behaves as a function of time. To do this we essentially use Newton's second law of motion: The mass times acceleration of each polymer is equal to the sum of forces acting on it. These forces include the force from all the other polymers around it, and from the molecules of the solvent. Acceleration is the rate of change of velocity, and velocity is the rate of change of position. So in order to apply Newton's second law we have to solve differential equations in order for us to find the position and velocity of each polymer as a function of time. This is where calculus enters the picture! The numerical solution of these equations with computers is usually called Molecular Dynamics Simulations.

Once we know how a system evolves in time, that is, once we know the position and velocity of every part of the system as a function of space and time, we can then calculate all the properties we are interested in. This methodology of finding the macroscopic properties of a system, as one would measure in a laboratory, from the behaviour of many molecules on a microscopic scale, is called Statistical Mechanics, which is a very beautiful subject, and a great achievement of mankind. Basically, one has to carry out averages of the properties of many many molecules, which involves numerical or analytical integration. Calculus again!

Often one is not interested in the motion of all the solvent molecules, but only in that of the polymers. The solvent molecules are then replaced with an effective force. Since they keep bombarding the polymer in random directions and at random times, the force is assumed to be a random noise. In this case, the differential equations for the polymer motion are no longer deterministic, but rather they become stochastic differential equations. The numerical integration of these equations needs entirely different methods from conventional calculus, and the field is called Stochastic Calculus. As you can imagine, the trajectory of each polymer is a random process in space, everywhere continuous but nowhere differentiable. Once we have integrated the stochastic differential equations, and we know the ensemble of trajectories for a system of polymers in time and space, we can again use Statistical Mechanics to solve for the properties of the solution as a whole. Numerically integrating the stochastic differential equations using computers is known as "Brownian dynamics simulations".

Calculus is undoubtedly an integral part of everything we do in our research.

Regarding Strogatz's claims regarding calculus: It is an exaggeration to say that calculus is at the heart of all scientific and technological progress, and to modern civilisation as we know it. It is definitely an integral part, and it is central to many of the achievements of mankind, but how does one separate one element of mathematics and give it the most prominent role? It would be more accurate to say that mathematics is central to modern society and all the technology and science we take for granted. Mathematics is the language with which we translate our mental construct of the world around us into a symbolic form that reveals the structure of the world and its order through the recognition of patterns. The representation of ideas in a symbolic form enables one to manipulate the symbols and make predictions, and this is the key to exploiting our understanding towards achieving practical outcomes. Mathematics has many branches, all of which seem necessary to create a symbolic representation of the world. Arithmetic, algebra, geometry, topology, logic . . . one could go on . . . surely there are many aspects of these subjects that do not have to do with the rates of change of variables, which is my understanding of calculus, and yet they are essential for the development of a mathematical framework.

Ravi Prakash Jagadeeshan is currently a Professor in the Department of Chemical Engineering at Monash University, Australia, where he heads the Molecular Rheology group. He has been at Monash since January 2001. Before joining Monash, Ravi was an Associate Professor at the Indian Institute of Technology, Madras, and did postdoctoral work on Sandpile dynamics with Prof. S. F. Edwards at Cavendish Laboratory in Cambridge, and on Polymer solution rheology with Prof. H. C. Öttinger at ETH Zürich. He was a Humboldt Fellow in the Techno-Mathematik Department at the University of Kaiserlautern in 1999/2000. His research is focussed on developing a theoretical and computational description of the flow behaviour of polymer solutions using a multiscale approach that combines molecular simulations at the mesoscopic scale with continuum simulations on a macroscopic scale. He is also interested in applying methods of soft matter physics to studying problems in biology.

III. Professor N. Mukunda

The problems I work on: My main interest has been in problems which are interesting from the physics point of view, in different areas which involve some amount of the mathematics of groups. This is a very important part of mathematics and it has overlap with various areas within mathematics and is also very important for problems in theoretical physics: classical mechanics, quantum mechanics, theoretical optics, statistical optics and quantum optics. All these areas have a group theoretical flavour. So I have been attracted to such problems. In addition I have looked at some examples of group representations which are suggested by physical ideas because they are in the same overall scheme.

On the use of calculus: Calculus comes into the picture among the groups that I have been interested in. That is the so-called continuous groups. There are other groups which are discrete, so the mathematics which goes with discrete groups is very different from that which we use when dealing with continuous groups. Continuous groups are also called Lie groups. For physical purposes the study of Lie groups involves calculus in a very basic way. This is one part of the answer. The other part is that in the second half of the 19th century there was a lot of work done in mathematics which led to what are called special functions of mathematical physics.These are solutions of important differential equations which arise in physical problems especially in the quantum mechanical framework. These functions are also closely related to group representations, I mean continuous group representation. All these ideas gel together: the use of differential equations based on calculus, the use of group theory for continuous groups, and applications of this in interesting physical problems.

Regarding Strogatz's claims regarding calculus: Calculus played a crucial role in understanding the mechanics of moving bodies starting with Newton. Years later it was crucial in understanding the laws of electromagnetism in Maxwell's classical theory. These were the basis of a lot of engineering and technology in the early part of the 20th century. Then when quantum mechanics came one of its versions was the wave mechanics of Schrödinger, and it turned out that that too was based on differential equations. So the mathematics based on calculus has been crucial for quantum physics, for classical electromagnetism, for the classical description of motion in all these fields. It is also important in the context of general relativity based on Reimannian geometry and other kinds of geometry. In fact, the calculus in that context is called the absolute differential calculus. You can see even by the choice of the name the importance of calculus in relativity. So there is little doubt that calculus is most important for practical understanding and uses in all these fields. I don't know much about how it is related to the human genome or the Declaration of Independence.

However, calculus by itself cannot lead us to the laws of nature, like Newton's equation of motion, or Maxwell's equations, or wave mechanics, or relativity. These are independent inputs from physics, to express which calculus is the most convenient language. So, the basic physical laws are not determined by calculus, but when discovered are expressed in the language of calculus. This distinction is very important.

Professor N. Mukunda got his PhD in physics from the University of Rochester, NY, USA in 1964. He worked at the Tata Institute of Fundamental Research, Bombay from 1959 to 1972 and the Indian Institute of Science from 1972 to 2001. His research interests are in quantum and classical mechanics, optics and theoretical physics.

IV. Professor Shobhana Narasimhan

The problems I work on: I work in the area of computational materials design. I use the techniques of a field called 'density functional theory' to understand why materials have he properties that they do. I then use this understanding to design novel materials that

possess desired properties for specific applications. I focus on nanomaterials, that are either structured at the nanometer scale or are composed of just a few atoms.

On the role of calculus: The basic equations in my field are known as the Kohn-Sham equations. They are differential equations, similar to the Schroedinger equation, and hence based on calculus. This is used for static properties. If one wants to solve for dynamic properties, such as the trajectory of an atom over time, we usually integrate Newton's equation of motion. "$F = ma$" may not immediately look like something that uses calculus, but since acceleration is the second derivative of position with respect to time, if one wants to find the position as a function of time, then if one knows the force at each time, one can find the position by integrating. I think the principle of calculus that I use most in my work is the idea that the minimum (or, more generally, extremum) of a function occurs where the first derivative of the function is equal to zero. I use this all the time in different ways. We use it to find the equilibrium geometry by searching for where the first derivative of the energy with respect to positions (which is equal to the force) is equal to zero. We also solve the Kohn-Sham equations by matrix diagonalization, and to diagonalize large matrices very fast on the computer, one can recast them as a minimization problem. We also solve for the density (modulus squared of wave function) of the system we are working on in a self-consistent way, and that too is recast as a minimization problem. To find the minimum using a computer algorithm, we use various numerical techniques, many of them are modifications of techniques meant to search for the zeros of a function (since finding the minimum of a function is the same as finding the zeroes of its first derivative). All this makes use heavily of ideas from calculus, even though something like matrix diagonalization may seem to be more of a problem from linear algebra.

Regarding Strogatz's claims regarding calculus: Well, I am not an expert in the history of science. But certainly very many of the fundamental equations of physics (Newton's equations, Maxwell's equations, the Schroedinger equations, etc.) are differential equations and hence make use of calculus. I do not know if there are alternative ways of solving these problems. I remember reading that though Newton solved many problems in classical mechanics using calculus, he kept his invention of calculus secret and, once he found the answer, re-derived the formulae using older techniques. I suppose it might be possible to do something analogous for some of the problems mentioned, but it would be incredibly tedious, and I am not sure how accurate it would be. The one I do not know what to make of is the claim that the Declaration of Independence would not have been made were it not for calculus! I am not sure what Strogatz had in mind. Maybe if people hadn't been able to calculate the trajectory of a cannonball using calculus, the outcome of some battle would have been different, thus changing the course of history?

Shobhana Narasimhan has a MSc in Physics from IIT Bombay, PhD in Physics from Harvard University, Postdocs at Brookhaven National Laboratory and Fritz Haber Institut, Berlin. Since 1996 on the faculty of the Theoretical Sciences Unit, Jawaharlal Nehru Centre for Advanced Scientific Research. Area of research is computational nanoscience, using density functional theory.

V. Professor Madan Rao

The problems I work on: I am a theoretical physicist working in the area of Statistical physics, both equilibrium and non equilibrium, and Soft Matter physics. Of late I have been engaged with understanding the physical principles underlying living systems across scales. We study the interplay between active mechanics, molecular organisation, geometry, and information processing in a variety of cellular contexts. We are interested in how living systems, composed of physical entities such as molecules and molecular aggregates, driven far from equilibrium, have self-organised (evolved) to perform engineering tasks, such as the efficient processing of information, computation, and control. This potentially brings together many fields of research, including nonequilibrium statistical physics, soft active mechanics, information theory, and control theory, to the study of biology.

On the role of calculus: Newton's math and its descendants pervade every aspect of my work and every physicist's work – Calculus, Real and Complex Analysis, Statistical physics and Probability/Information theory, Approximation and Numerical analysis, Dynamical systems, Differential geometry of curves and surfaces, and Partial and Ordinary Differential equations.

Regarding Strogatz's claims regarding calculus: There is a consensus amongst mathematicians that Newton is amongst the top 10 mathematicians of all times, and I do remember a poll that declared that Newton was at the top of this list (I am afraid I can't recall who conducted the poll, possibly AMS?). (If I recall correctly, the list has Archimedes, Aryabhata, Newton, Euler, Gaüss, . . .) Why is this so? In my opinion there are three parts to the answer – Natural Philosophy, Mathematics and as a Language of Science (Physics). Natural Philosophy: Briefly, Aristotelian natural philosophy posited that nature could be comprehended by rational thought, by the application of hard logic. Newtonian philosophy went beyond this and asserted that nature could not only be comprehended, but that nature was, in principle, calculable and hence predictable, using the language of mathematics. Mathematics: While Newton's contribution to mathematics is many fold, Calculus is indeed one his priceless contributions. This includes the notion of infinitesimals, the notion of limits – through which one may get at hidden features of functions. Newton then systematically developed the ideas of Differentiation, Integration and the fundamental theorem of calculus. This leads directly to Analysis (...Jacobi, Bernoulli), Non Euclidean geometries, (Differential) geometry of curves and surfaces, higher dimensional spaces (..Gauss, Riemann) that brings together geometry and analysis, Series expansions (...Euler, Jacobi), Approximations and numerical analysis (......Gauss), Differential equations – local and global analysis leading to dynamical systems, Vectors / Tensors and their analysis, Continuous distributions and probability theory, Measure theory, ...and many more. Almost the whole edifice of mathematics.

Physics: Newton heralded a period which saw an intimate relation between physics and math. His deep philosophical insight, standing on the shoulders of Galileo, was to see and make precise Kinematics – the nature of space and time – as the underlying foundation on which a Mechanics of the Movement of Bodies could be built. With the relativity principle and inertial frames, Galileo and Newton potently refuted Aristotelian natural philosophy. Galileo however could make progress in rectilinear movement and simple curvilinear movement. The general motion of bodies needs the concept of instantaneous velocity, which in turn can only be defined by a limiting procedure and differential calculus – the hidden mathematical kernel that was gleaned by Newton. Newton's proofs combine the calculus he discovered with geometry, and are considered a thing of beauty (see book by S. Chandrasekhar). Newton's creative mind shed light on every aspect of the natural (non-living) world that he could perceive at that time.

Madan Rao is a theoretical physicist at the Simons Centre for the Study of Living Machines, National Centre for Biological Sciences (TIFR), Bangalore and affiliated with the International Centre for Theoretical Sciences (TIFR). He received his Ph.D from Indian Institute of Science, Bangalore in 1989 and post-doctoral training at Simon Fraser University, Vancouver. Subsequently, he held positions at the Institute of Mathematical Sciences, Chennai and Raman Research Institute, Bangalore. Madan works in the areas of nonequilibrium statistical physics, and soft and biological physics. At the Simons Centre, Bangalore, he has helped build a vibrant centre for Theory in Biology.

VI. Professor Prajval Shastri

The problems I work on: I investigate how giant black holes that are in the centres of galaxies like our Milky Way behave, by imaging their environs using telescopes. A black hole that is over a million to several billion times the mass of our Sun typically inhabits the centres of galaxies, even to very early cosmic times. They draw in any matter that approaches very close to them, accelerating it enormously and swirling it in, and consequently this matter is heated to enormous temperatures to become a beacon in the sky. These beacons are picked up by telescopes on earth even from billions of light years away. We can therefore image them, how they change, how they influence the evolution of their host galaxies, and also how they are formed.

On the role of calculus: Calculus is so integral to a physicist's thinking that most often we are not even consciously thinking "I am using calculus here". Mathematics enables us to describe the phenomena that we observe and measure, in a self-consistent language, that in turn enables us to both unearth something deeper than what an individual phenomenon suggests, and anticipate phenomena that we can observe and measure (or not, and thereby refute the earlier descriptions). In my own work, calculus plays a role in the tools used as well. For example astrophysicists use a technique of measuring the Fourier components of the Fourier transform of a piece of the sky, in order to reconstruct the original image at a level of sharpness that would otherwise be impossible, which is akin to constructing the sound of a flute from a whole bunch of "pure" or "tuning-fork-type" notes. Calculus plays a key role in this technique.

Regarding Strogatz's claims regarding calculus: Exceptionalizing calculus in technological developments unnecessarily undermines the importance of tinkering and trial-and-error in engineering. For example, while GPS technology would not work without taking into account predictions of General Theory of Relativity, in a hypothetical situation where we had no General Relativity but only satellite-borne transmitters and receivers, it is quite possible that the need to offset transmitted and received frequencies (i.e., the General Relativistic offset) might have been arrived at by mere tinkering.

Knowledge creation is a process of building upon and constantly testing, interrogating and reaffirming (or not) past knowledge, and is perforce a collective, co-operative human endeavour. Therefore to create straw-persons that privilege some particular branch or toolkit that has contributed to the current positive aspects of the human condition becomes merely polemics. Furthermore, the extent to which any particular insight from any discipline realises its potential by impacting global knowledge creation is heavily dependent on the socio-political context in which the insight emerged. Therefore, in evaluating the role of mathematics to our current understanding of the world, it is far more useful to (a) evaluate the paths that specific insights took in their socio-political-historical context, and (b) foreground the pitfalls of how privileging mathematics can lend false credibility to predictive mathematical modelling, especially of environmental, public health, economic and other complex systems critical to human well-being, when the modelling could be fundamentally flawed because of flawed assumptions.

Prajval Shastri is an astrophysicist, formerly at the Indian Institute of Astrophysics, Bengaluru. Her core research interest lies in empirical investigations of supermassive black holes. Her core angst is discontent with early science education. Acutely aware of her privilege of being paid to be fascinated by the universe, Prajval is continually amazed that amateur astronomers often stand out as the impassioned ones. She can be contacted at prajval.shastri@gmail.com.

VII. Avehi Singh

The problems I work on: I am studying the ways in which bees find flowers using their sense of smell. Flowers emit blends of multiple volatile chemicals that have been shown to attract insects to them. For my work, I am testing whether individual chemicals from flower scents are detected by the brains of bees. Information is passed through the insect brain in the form of small electrical pulses (not exceeding 70 millivolts or 0.07 volts). By measuring the voltage change across the antenna (this is the primary organ of smell in insects) in response to a particular chemical from a flower, I identify chemicals that may be helping bees find these flowers. For example, if a chemical induces a particular voltage change in the antenna, I conclude that the insect brain is detecting it and can subsequently test whether this chemical produces any behavioural response in the bees.

On the role of calculus: A main challenge of using this technique (called electrophysiology) is differentiating between signal and noise. The voltage fluctuations caused by brain activity are very small (for reference, the voltage across an ordinary light bulb is 120 volts) and transient. A sensitive electrode can detect changes in voltage caused by humidity, static and any number of other random factors, which make it hard to identify brain activity. In order to determine whether a voltage change is caused by brain activity and not noise, we use mathematical models that can describe these electrical pulses. A main class of these models are the Hodgkin-Huxley models, which are a series of ordinary differential equations that describe the flow of current into and out of brain cells during brain activity. By using these equations, we can parse the output from the antenna into signal and noise.

Avehi Singh is a first-year PhD student in the Ecology program at Penn State University. She has a bachelor's degree in Biology from Reed College, Oregon. Her PhD research is on the evolution of sensory systems in pollinating insects and she is interested in interdisciplinary research questions that span biology, chemistry and mathematics. She attributes her interest in science to her schooling in Bangalore, where she spent many hours exploring the beautiful deciduous forests around the city. She enjoys hiking, cooking and spending time with her dogs.