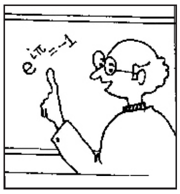
Classroom



In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Optimal Extraction of Heat: An Instructive Problem*

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1. Introduction

In this article, we pose a challenge concerning an optimal algorithm for transferring heat from a hot liquid to a cold one. The solutions discussed here demonstrate the usefulness of choosing the right variables. We illustrate different possible approaches—algebra, calculus, and numerical simulation–each of which has its charm. Instructors can make good use of this to illustrate multiple methods of problem-solving in an undergraduate classroom.

One of us (JVP) first learnt of this challenge in a short course conducted by Prof. G. S. Ranganath of the Raman Research Institute at Research Education Advancement Programme, J. N. Planetarium, Bengaluru in 2007.

2. Challenge

There are two conducting metal vessels of negligible mass and perfect conductivity, each containing water of mass M_0. One is closed and is at a higher temperature T_H. The other is open and is at a lower temperature T_C. What is the maximum temperature (T_M) to which the water in the open container be taken by strategically bringing it in contact with the hotter one? You have as many other similar empty open vessels as you wish, and there are

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no losses like that due to radiation, evaporation, or conduction, except between the hot vessel and the colder vessels.

2.1 *Principle of Calorimetry*

If mass m_A of water at temperature T_A is brought into contact with mass m_B at temperature T_B (the specific heats being the same), then the resultant temperature T_R is given by the conservation of energy as:

In case different liquids are used, the respective specific heat capacities have to be included.

$$m_A s(T_A - T_R) = m_B s(T_R - T_B), \tag{1}$$

where s is the specific heat capacity of water. This is the principle of calorimetry. This equation simplifies to

$$T_R = \frac{m_A T_A + m_B T_B}{m_A + m_B}. \tag{2}$$

If we bring the two vessels straight into contact, then the resultant temperature of both the vessels would be $(T_H + T_C)/2$. Can we do better than that?

2.2 *Solution*

Here is a strategy: Let us distribute the water in the open container *equally* into N containers; bring them, one by one, into contact with the closed vessel; and then mix all N of them.

Keeping track of the temperature of the closed vessel is easier. Conservation of energy then helps us calculate the temperature of the open water mixture.

If T_i is the temperature of the closed vessel after i contacts, then from (2),

$$T_{i+1} = \frac{(M_0 \times T_i) + \left(\frac{M_0}{N} \times T_C\right)}{M_0 + \frac{M_0}{N}} = \frac{NT_i + T_C}{N+1}. \tag{3}$$

Clearly, $T_0 = T_H$ (initial temperature). Hence, by (3),

$$T_1 = \left(\frac{N}{N+1}\right) T_H + \frac{T_C}{N+1}. \tag{4}$$

Applying (3) successively and extending,

$$\begin{aligned}
T_2 &= \frac{NT_1 + T_C}{N+1} = \left(\frac{N}{N+1}\right)^2 T_H + \frac{T_C}{N+1}\left[\left(\frac{N}{N+1}\right)^1 + 1\right] \\
T_3 &= \frac{NT_2 + T_C}{N+1} = \left(\frac{N}{N+1}\right)^3 T_H + \frac{T_C}{N+1}\left[\left(\frac{N}{N+1}\right)^2 + \left(\frac{N}{N+1}\right)^1 + 1\right] \\
&\vdots \\
T_N &= \left(\frac{N}{N+1}\right)^N T_H + \frac{T_C}{N+1}\left[\left(\frac{N}{N+1}\right)^{N-1} + \left(\frac{N}{N+1}\right)^{N-2} + \cdots + \left(\frac{N}{N+1}\right)^1 + 1\right].
\end{aligned}$$

This geometric series has a finite number of terms, each of which is finite, and is easily summed using the standard formula to give

$$\begin{aligned}
T_N &= \left(\frac{N}{N+1}\right)^N T_H + T_C\left[1 - \left(\frac{N}{N+1}\right)^N\right] \\
&= T_C + (T_H - T_C)\left(\frac{N}{N+1}\right)^N. \qquad (5)
\end{aligned}$$

The case of $N = 1$ is same as bringing the two vessels straight into contact with each other and (5) reproduces the expected result $T_C + (T_H - T_C)/2 = (T_H + T_C)/2$. The $N = 2$ case decreases the closed vessel temperature further to $T_C + (T_H - T_C)/2.25$, which is promising. The factor $(N/(N+1))^N$ keeps decreasing as N increases, which can be verified by considering the ratio for two successive values of N. Therefore, taking the limit $N \to \infty$, and using the well known result $\lim_{N\to\infty} (1 + 1/N)^N = e \approx 2.71828$,

$$T_\infty = T_C + \frac{T_H - T_C}{e}. \qquad (6)$$

Hence, by conservation of energy, the maximum temperature, T_M, to which the open water mixture can be raised, by this algorithm, would be

$$T_M = T_H + T_C - T_\infty = T_H - \frac{T_H - T_C}{e}. \qquad (7)$$

For example, if $T_H = 80^\circ$C, $T_C = 20^\circ$C, then $T_M = 57.9^\circ$C. Note that the temperature of the originally hot vessel will now be 42.1°C!

We can simplify the algebra above if we are careful about the zero point of the temperature scale. So, without losing any generality, we can take $T_C = 0$. Then, from (3),

$$T_{i+1} = \frac{NT_i}{N+1}, \tag{8}$$

which means, there is a multiplication by a factor $(N/(N+1))$ at each stage. With $T_0 = T_H$, this gives

$$T_N = \left(\frac{N}{N+1}\right)^N T_H, \tag{9}$$

which, in the old temperature scale, recovers (5) (Think it over!).

3. A Follow-up Challenge

Why is dividing the mixture *equally* the best strategy?

3.1 *Solution A*

We first adopt a numerical approach, where the best strategy is chosen by improving the heat extraction by successively changing the mass distribution. We introduce this straightforward and powerful computational method through the following algorithm:

- Begin with an unequal distribution of water in different vessels.
- Calculate the temperature of the final mixture.
- Try to slightly change the distribution randomly and accept the change if and only if the temperature of the final mixture is going to increase.
- Keep repeating this process until the distribution converges.

In *Figure* 1, a particular distribution is chosen with unequal amounts of water in different vessels (black curve). The algorithm mentioned above drives this unequal distribution to an equal distribution (red curve). The number of iterations is kept sufficiently large to ensure convergence. Note that this may not be computationally the most efficient algorithm. But, being random, it does not make use of any additional knowledge about the problem.

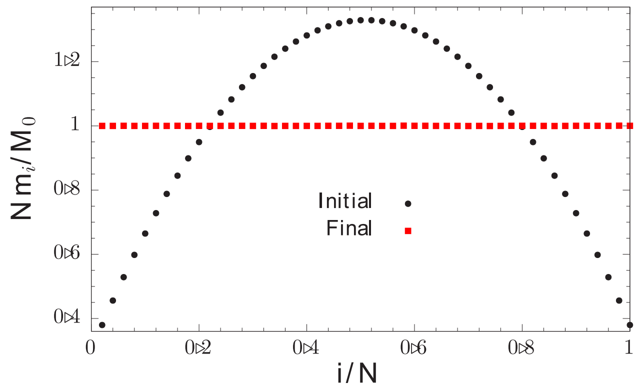

Figure 1. The variational solution to the follow-up challenge. The total number of vessels, $N = 50$. i is the vessel number, and m_i is the mass of water it holds. 10^5 iterations were performed. Unequal initial distribution is taken to an equal final distribution.

3.2 *Solution B*

This approach provides intuition as to why equal distribution is the best strategy. First take $N = 2$. Let the first vessel carry water of mass x and the second $(M_0 - x)$. In the temperature scale used earlier, where $T_C = 0$, it follows that

$$T_1 = \frac{T_H}{1 + \frac{x}{M_0}} \qquad \text{and} \qquad T_2 = \frac{T_H}{\left(1 + \frac{x}{M_0}\right)\left(1 + \left(\frac{M_0 - x}{M_0}\right)\right)}. \tag{10}$$

Clearly, T_2 is minimized when $x = M_0/2$ or when the water is distributed equally between the two vessels. A similar argument shows that equal division works best even if the total amount of water in the cold vessels is different from M_0. We will use this general result soon.

Now, for a general N, consider a distribution where not all vessels contain the same amount of water. This means there has to exist at least one pair of successive vessels which contain unequal amounts of water. From the analysis just done for the case of $N = 2$, we know that redistributing the water between these two vessels until they have equal amounts of water is going to give us a better heat extraction by the pair of vessels. So let us equalize them. If we keep repeating this process, clearly, we will end up with an equal amount (M_0/N) of water in all the N vessels as the best solution. Now the students may see how intuition is typically connected with equations.

3.3 *Solution C*

This is an analytical approach to the same problem. If there are N open vessels, and m_i is the mass of water in the i^{th} vessel, then, it is a good exercise for the student to generalize (10), which was applicable for $N = 2$ case, and check that the temperature of the closed vessel (in the scale where $T_C = 0$) after all the N contacts would be

The base of the logarithm is e everywhere.

$$T_N = \frac{T_H}{\prod_{i=1}^{N}\left(1 + \frac{m_i}{M_0}\right)} = T_H \exp\left(-\sum_{i=1}^{N} \log\left(1 + \frac{m_i}{M_0}\right)\right), \qquad (11)$$

which is to be minimized by varying the m_is, subject to the constraints

$$\sum_{i=1}^{N} \frac{m_i}{M_0} = 1 \qquad \text{and} \qquad 0 \leq \frac{m_i}{M_0} \leq 1. \qquad (12)$$

Physically, this condition means that each of the vessel must contain a nonnegative mass of water and the sum of all such masses has to be M_0. So, among all the possibilities that respect these constraints, we are looking for the best one. We can, again, proceed further in two ways.

3.3.1 *Algebraic method*

Motivated by the form in which the log function appears in (11), we recall the well known inequality,

It is an instructive exercise to plot both the functions to visualize this inequality.

$$\log(1 + x) \leq x \qquad (\text{equality for } x = 0). \qquad (13)$$

Applying the inequality (13) in (11) and then making use of (12), one can conclude that $T_N \geq T_H/e$. In other words, the temperature of the closed bin can never get below T_H/e, irrespective of N. In view of (6), since the equal distribution of m_is, in the limit $N \to \infty$ achieves this temperature, no other strategy can do better than that.

3.3.2 *Optimization using Lagrange multipliers*

Extremizing a function subject to constraints can be elegantly done using Lagrange's method of undetermined multipliers (see [1]). Here, from (11), it is enough to minimize

$$F(m_1, \ldots, m_N) = \sum_{i=1}^{N} -\log\left(1 + \frac{m_i}{M_0}\right), \tag{14}$$

subject to the constraints given in (12). This is accomplished by setting

$$\frac{\partial}{\partial m_j} \sum_{i=1}^{N} \left(-\log\left(1 + \frac{m_i}{M_0}\right) + \lambda \frac{m_i}{M_0}\right) = 0\,, \tag{15}$$

where λ is the Lagrange multiplier. This reduces to $1/\left(1 + \frac{m_j}{M_0}\right) = \lambda$, implying that all the m_js are equal. Eqn. (12) then gives $m_j = M_0/N$.

This solution certainly corresponds to a constrained extremum. Investigating the nature of the extremum in the general case is elaborate [2]. But in this case, the investigation can be simplified greatly as follows.

We first note that F is a sum of terms, each of which is a function of a single variable. This allows us to expand F in the Taylor series about the solution obtained, up to quadratic order as

Technically, this means $\frac{\partial^2 F}{\partial m_i \partial m_j} = 0$, if $i \neq j$.

$$\begin{aligned} F(m_1, \ldots, m_N) - F\left(\frac{M_0}{N}, \ldots, \frac{M_0}{N}\right) &= -\frac{N}{(N+1)M_0} \sum_i \left(m_i - \frac{M_0}{N}\right) \\ &+ \frac{1}{2}\left(\frac{N}{(N+1)M_0}\right)^2 \sum_i \left(m_i - \frac{M_0}{N}\right)^2 . \end{aligned}$$

Further, the linear term above vanishes because of the constraint (12). Hence, for m_is that are consistent with the constraint, the RHS is always positive, ensuring that F and thereby T_N is a constrained local minimum at the solution obtained. Also, since a unique solution was obtained to (15), we can conclude that this is a constrained absolute minimum of T_N.

4. Continuum Limit

Working in the continuum limit involves a subtle aspect. Let us calculate the final temperature when the cold water slowly trickles on the closed vessel 'drop by drop'.

Continuing to work in the temperature scale where $T_C = 0$, if the mass of each drop is δm, then from (2), the change in temperature of the closed vessel will be

$$\delta T = \frac{(M_0 \times T) + (0 \times \delta m)}{M_0 + \delta m} - T \approx -T\,\frac{\delta m}{M_0}\,, \qquad (16)$$

where m goes from 0 to M_0 in a continuous fashion. The approximation made in (16) works better and better for smaller and smaller $\delta m/M_0$. In the limit $\delta m/M_0 \to 0$, (16) reduces to the differential equation

$$d \log T = -\frac{dm}{M_0}. \qquad (17)$$

With the initial temperature of the closed vessel as T_H, this solves to $T = T_H e^{-(m/M_0)}$. So the end temperature of the closed vessel is T_H/e. Going back to the original temperature scale, this will recover (6).

This analysis means that as long as the mass of the biggest drop goes to zero in the continuum limit, the same final temperature will be attained by the mixture. The mass of each drop need not be the same!

To check this, we keep the 'shape' of the initial distribution to be the same as that given in *Figure* 1 and gradually change N. So we have unequal drops at every stage of subdivision, with the most massive drop around 3.5 times heavier than the least massive one. For each N, we calculate the final temperature T_N attained by the closed vessel. T_N monotonically decreases with N and as $N \to \infty$, $T_N/T_H \to 1/e$, in agreement with the continuum analysis. This can also be seen in *Figure* 2.

To further check our understanding, we give a nonrigorous estimate for the difference between the continuum answer and the

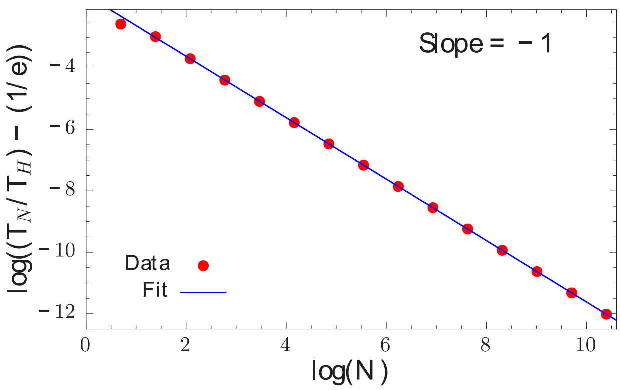

Figure 2. Approaching the continuum limit by gradually increasing N. T_N is the temperature of the closed vessel. T_H is its initial temperature. N here stands for the number of drops.

answer at any finite stage. We obtain the rate of convergence to the continuum limit in the case when the subdivisions are not equal. If the difference in the masses of two successive drops is determined by a scale ε, then ε decreases as $1/N$ when N increases. Since the order of the two drops will not affect the final temperature of the closed vessel, we can rule out an error of order ε. We, therefore, take the leading dependence on ε, of the deviation from the continuum limit to be ε^2, which goes as $1/N^2$ when N increases. Since there are N drops, the above-mentioned deviation, summed over all the drops, should decrease as $1/N$ when N increases. Sure enough, the graph in *Figure* 2 approaches a straight line for large N and its slope is -1.

5. Industry Perspective

Heat exchange processes are of great practical importance. For example, milk is pasteurized by heating for a short time. To avoid wastage of energy, the yet to be pasteurized milk flowing into the heating chamber is placed in thermal contact with the outflowing pasteurized milk. This way, the only heat which is to be supplied is to make up for the losses from the system. This is a more general case than the one we have considered since the hot fluid itself has regions with different temperatures. Students with an engineering bent of mind can consult [3]. The general point of achieving the goal in multiple stages continues to be valid. More-over, various approximations, as stated at the beginning itself, impact the problem, and are to be accounted for in a 'real-life' situation. Clearly, we have milked this problem for multiple classroom applications!

Suggested Reading

[1] G B Thomas and R L Finney, *Calculus and Analytic Geometry*, Pearson Education, 9th edition, Sec. 12.9, pp.998–1007, 2005.
[2] https://sites.math.northwestern.edu/~clark/285/2006-07/handouts/lagrange-\protect\@normalcr\relax2deriv.pdf
[3] https://en.wikipedia.org/wiki/Plate_heat_exchanger