Problem Corner

Adventures in PROBLEM SOLVING

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In this edition of 'Adventures' we study a few miscellaneous problems, some from the PRMO and some from the AIME (the 'American Invitational Mathematics Examination'). As usual, we pose the problems first and present the solutions later.

Miscellaneous problems

Problem 1. Let a, b be natural numbers such that $2a - b$, $a - 2b$ and $a + b$ are all distinct squares. What is the smallest possible value of b? (Problem 15, PRMO 2018)

Problem 2. Raju's age and his father's age in years are 2-digit integers. When the father's age is written after Raju's age, a 4-digit perfect square is formed. If the father's age 25 years ago is written after Raju's age at that time, another 4-digit perfect square is formed. What are the ages of Raju and his father? (Problem shared with me over email; thank you, Hitha.)

Problem 3. A 5-digit number n is such that when its middle digit is removed, the resulting 4-digit number m is a divisor of n. Find all possible values of n/m. (Purdue, "Problem of the Week")

Problem 4. (a) What is the largest prime factor of the binomial coefficient $\binom{2000}{1000}$?

(b) What is the largest 2-digit prime factor of the binomial coefficient $\binom{200}{100}$? (AIME 1983)

Keywords: Binomial coefficient, prime factor, subset, Pythagorean triple

Problem 5. For each non-empty subset of $\{1, 2, 3, 4, 5, 6, 7\}$, arrange the members in decreasing order with alternating signs and take the sum. For example, for the subset $\{5\}$ we get 5. For $\{6, 3, 1\}$ we get $6 - 3 + 1 = 4$. Find the sum of all the resulting numbers. (AIME 1983)

Solutions to the problems

Solution to problem 1

The problem requires us to look for pairs (a, b) of natural numbers such that $2a - b$, $a - 2b$ and $a + b$ are distinct squares. Let $2a - b = x^2$, $a - 2b = y^2$ and $a + b = z^2$. Since $y^2 \geq 0$, it follows that $a \geq 2b$. If $a = 2b$, then $x^2 = 3b$ and $z^2 = 3b$, so x^2 and z^2 are not distinct squares (contrary to the given information). Hence $a > 2b$, and x^2, y^2 and z^2 are non-zero squares. We now reason as follows.

- Since $(2a - b) - (a - 2b) = a + b$, i.e., $x^2 = y^2 + z^2$, (y, z, x) is a Pythagorean triple.
- We recall the following from number theory: *under division by 3, a square number leaves remainder 0 or 1; the remainder is never 2*. Here are two implications of this: (i) if the sum of two squares is a multiple of 3, then both squares are multiples of 3; (ii) in a Pythagorean triple, at least one number in the triple is a multiple of 3.
- Observe that $x^2 + y^2 = 3a - 3b$, which is a multiple of 3. Hence both x and y are multiples of 3. Therefore z too is a multiple of 3. So 3 divides all three numbers in the triple (y, z, x). This tells us that $(y/3, z/3, x/3)$ too is a Pythagorean triple.
- The problem asks us to find pairs (a, b) of natural numbers satisfying the stated conditions, with b as small as possible. This amounts to finding a Pythagorean triple (y, z, x) in which all three numbers are multiples of 3, with z^2 and y^2 as close to each other as possible.
- It is natural to start by looking at primitive Pythagorean triples in which the two smaller numbers are small and as close as possible. Since no Pythagorean triple exists in which the least number is ≤ 2 (please verify this for yourself), we focus on the triple $(3, 4, 5)$. We clearly cannot do better than this. This means that in any Pythagorean triple, the smallest number is ≥ 3 and the second smallest number is ≥ 4, so the sum of the two smaller numbers is ≥ 7 and the difference between the squares of the two smaller numbers is ≥ 7.
- Since $(y/3, z/3, x/3)$ is a Pythagorean triple, it follows that $(z/3)^2 - (y/3)^2 \geq 7$, and therefore that $z^2 - y^2 \geq 63$. Hence $b \geq 21$.
- Now it is easy to check that the value $b = 21$ can be 'realised.' We start with the triple $(3, 4, 5)$ and scale it up by a factor of 3; we get $(9, 12, 15)$. The squares of these three numbers are 81, 144, 225 respectively. So we write $a - 2b = 81$, $a + b = 144$ and solve for a, b; we get $a = 123$, $b = 21$. For this choice of values for a and b, the numbers $2a - b$, $a - 2b$ and $a + b$ are indeed distinct squares; please check. This justifies the claim that the value $b = 21$ can be realised. Hence the least possible value of b is 21.

Solution to problem 2

Let the son's age be a years, and let the father's age be b years; both a and b are two-digit numbers. As per the data given, the following can be stated:

- the number $100a + b$ is a perfect square;
- the number $100(a - 25) + (b - 25)$ is also a perfect square.

Let $100a + b = u^2$ and $100(a - 25) + (b - 25) = v^2$. Since u^2 and v^2 are 4-digit perfect squares, u and v are 2-digit numbers. This means that $u + v < 200$.

By subtraction, we get $u^2 - v^2 = 2525$. This can be written as $(u + v)(u - v) = 2525$.

Now the number 2525 can be factorised as $2525 = 25 \times 101$. Importantly, 101 is a prime number. In what ways can 2525 be written as a product of two numbers, neither of which exceed 200? Precisely because 101 is a prime number, the only possible way is 25×101. This implies that $u - v = 25$ and $u + v = 101$. By addition we obtain $2u = 126$, hence $u = 63$ and therefore $v = 38$.

Hence $u^2 = 63^2 = 3969$ and $v^2 = 38^2 = 1444$. So the son's age is 39 and the father's age is 69 (current ages).

Solution to problem 3

Let a be the 2-digit number formed by the leftmost two digits of n; let b be the middle digit; and let c be the 2-digit number formed by the rightmost two digits of n. Then we have

$$n = 1000a + 100b + c, \qquad m = 100a + c.$$

Since m is a divisor of n, we have

$$\begin{aligned} & 100a + c \mid 1000a + 100b + c, \\ \therefore\ & 100a + c \mid 1000a + 100b + c - 10(100a + c), \\ \therefore\ & 100a + c \mid 100b - 9c. \end{aligned}$$

Now we establish some inequalities. We compute the least possible value of $100a + c$ and the greatest possible (absolute) value of $100b - 9c$. (Note that $100b - 9c$ can be negative.) We have

$$100a + c \geq 100 \times 10 = 1000,$$

i.e., $100a + c \geq 1000$. Also, since $0 \leq b \leq 9$ and $0 \leq c \leq 99$, we have

$$-99 \times 9 \leq 100b - 9c \leq 100 \times 9,$$

i.e.,

$$-891 \leq 100b - 9c \leq 900.$$

Since $100b - 9c$ is required to be a multiple of $100a + c$, the only way this can happen is for $100b - 9c = 0$, which in turn can only happen if $b = 0$ and $c = 0$. But if these conditions hold, then $n = 1000a$ and $m = 100a$, and the requirement that $m \mid n$ is automatically met, for any value of a.

In each case we find that $n/m = 10$; so this is the required answer.

Solution to problem 4 (a)

It should be fairly clear that the number

$$\binom{2000}{1000} = \frac{2000 \times 1999 \times 1998 \times 1997 \times \cdots \times 1003 \times 1002 \times 1001}{1 \times 2 \times 3 \times 4 \times \cdots \times 998 \times 999 \times 1000}$$

is divisible by every prime number between 1001 and 2000 (for this prime number will be a factor of the numerator of the above expression but not a factor of the denominator). Hence the answer is simply the largest prime number between 1001 and 2000. It so happens that 1999 is a prime number. Hence this is the desired answer.

Solution to problem 4 (b)
We require the largest two-digit prime factor of the number

$$\binom{200}{100} = \frac{200 \times 199 \times 198 \times 197 \times \cdots \times 103 \times 102 \times 101}{1 \times 2 \times 3 \times 4 \times \cdots \times 98 \times 99 \times 100}.$$

Here is a list of all the two-digit prime numbers:

$$11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 83,\ 89,\ 97.$$

We start from the 'upper end.' Could 97 be the answer? No, because it is present once in the denominator (as 97 itself), and once in the numerator as well (as a factor of 194). So 97 'cancels' out, which means that the number $\binom{200}{100}$ is not divisible by 97. Could 89 be the answer? Arguing as earlier, we note that it is present once in the denominator (as 89 itself) and once in the numerator (as a factor of 178), so $\binom{200}{100}$ is not divisible by 89. Could 83 be the answer? Yet again the answer is no, for the same reason. We see a way forward now. We clearly require the largest two digit prime number p such that *there are more multiples of p between* 101 *and* 200 *than between* 1 *and* 100. This is equivalent to searching for the largest prime number p such that $p < 101 < 2p < 3p < 200$. (This way there are two multiples of p between 101 and 200, and only one multiple of p between 1 and 100.) The number which fits this description is 61 (note that $67 \times 3 = 201 > 200$); hence 61 is the required answer.

Using the same reasoning, we can show that $\binom{200}{100}$ is divisible by each of the following two-digit primes: 59, 53, 37, 17, 13, 11. In the case of 37, we find that it is present twice in the denominator and three times in the numerator. Observe that the primes 41, 43 and 47 are missing from the list. Each of these occurs twice in the denominator and twice in the numerator, thereby canceling out.

Here is the full expression of $\binom{200}{100}$ as a product of primes:

$$\begin{aligned}\binom{200}{100} &= 2^3 \times 3 \times 5 \times 11 \times 13^2 \times 17 \times 37 \times 53 \times 59 \times 61 \times 101 \times 103 \times 107\\ &\quad \times 109 \times 113 \times 127 \times 131 \times 137 \times 139 \times 149 \times 151 \times 157 \times 163\\ &\quad \times 167 \times 173 \times 179 \times 181 \times 191 \times 193 \times 197 \times 199.\end{aligned}$$

It is curious that only two primes occur to a power greater than 1; namely, 2 and 13.

Solution to problem 5
This is a truly beautiful problem!

In an effort to get a handle on the problem, we start with smaller sets and build our way upward, all the while looking for a pattern. (We can do this using hand calculation.) Here is what we find.

Set	Sum
$\{1\}$	1
$\{1,2\}$	4
$\{1,2,3\}$	12
$\{1,2,3,4\}$	32

A pattern is already becoming evident: *it appears that if the set is* $\{1,2,3,4,\dots,n\}$*, then the sum is* $n \cdot 2^{n-1}$. If this pattern is valid, then the required answer is $7 \cdot 2^6 = 448$.

But what could be the explanation for the sum to have this simple form? To find it, we go back to the original problem, in which the largest number is 7.

Consider any subset A of $\{1, 2, 3, 4, 5, 6, 7\}$ such that $7 \in A$. Let B be the subset obtained by removing 7 from A, i.e., $B = A \setminus \{7\}$. Let the alternating sums associated with sets A and B, computed the way described in the problem, be a and b respectively. What will $a + b$ be equal to? A moments reflection reveals that the answer must be 7. To see why, we look at a simple instance. Suppose that $A = \{2, 5, 7\}$ and $B = \{2, 5\}$. Then the alternating sums associated with the two sets are $a = 7 - 5 + 2$ and $b = 5 - 2$ respectively. Adding them, we see that a beautiful cancellation takes place, and the sum is 7.

We infer from this phenomenon that the sum of all such alternating sums is equal to 7 times the number of subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ not containing the element 7, i.e., the number of subsets of $\{1, 2, 3, 4, 5, 6\}$. This number is $2^6 = 64$. Hence the required sum is $7 \times 64 = 448$.

Generalising, if the initial set is $\{1, 2, 3, \ldots, n\}$, the required sum is $n \cdot 2^{n-1}$.

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