ClassRoom

Finding the Square Root of a Four-Digit Perfect Square

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On the web page [1] belonging to *A2Y Academy For Excellence*, there appears a curious method to find the square root of a four-digit number if it is known that that number is a perfect square. We describe the method here using two examples and then consider how to explain it.

Example 1. To find the square root of 3969 (assuming that this number is a perfect square). We proceed as follows.

- Split the given number into two blocks of two digits each; here we obtain 39 and 69. Call these the 'first' number and the 'second' number, respectively.
- Consider the first number, 39. The largest perfect square less than or equal to 39 is $36 = 6^2$. Inference: the 10's digit of the square root is 6.
- Now consider the second number, 69. Since its 1's digit is 9, the 1's digit of the square root is either 3 or 7. To figure out which one, we proceed as follows.
- We compute the product $6 \times 7 = 42$ (the '6' here is the 10's digit of the square root, and 7 is the integer after 6, i.e., $7 = 6 + 1$) and compare this with the first number, 39. Since $39 < 42$, we select the *smaller* of the two choices $(3, 7)$, i.e., we choose 3. Inference: the 1's digit of the square root is 3.
- Hence the desired square root is 63. (Check: $63^2 = 3969$.)

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Example 2. To find the square root of 5776 (assuming that this number is a perfect square). We proceed as follows.

- Split the given number into two blocks of two digits each; we obtain 57 (the 'first' number) and 76 (the 'second' number).
- Consider the first number, 57. The largest perfect square less than or equal to 57 is $49 = 7^2$. Inference: the 10's digit of the square root is 7.
- Now consider the second number, 76. Since its 1's digit is 6, the 1's digit of the square root is either 4 or 6.
- We compute the product $7 \times 8 = 56$ (the '7' here is the 10's digit of the square root, and 8 is the integer after 7, i.e., $8 = 7 + 1$) and compare this with the first number, 57. Since $57 > 56$, we select the *larger* of the two choices, i.e., we choose 6. Inference: the 1's digit of the square root is 6.
- Hence the desired square root is 76. (Check: $76^2 = 5776$.)

The method described abstractly. The steps can be stated compactly as follows. The task is to find the square root of a given number N, $10^2 \leq N < 10^4$, if it is known that N is a perfect square. Let $N = 100a + b$, where $1 \leq a < 100$ and $0 \leq b < 100$. Let the desired square root be written as $10c + d$ where $1 \leq c < 10$ and $0 \leq d < 10$ (so c and d are single-digit numbers).

1. We find c from $c = \lfloor \sqrt{a} \rfloor$. (Here the symbol $\lfloor\ \rfloor$ denotes the 'floor function' defined as follows: $\lfloor z \rfloor =$ the greatest integer that does not exceed z; e.g., $\lfloor 3.7 \rfloor = 3$, $\lfloor 1.9 \rfloor = 1$.)
2. From the 1's digit of b, we deduce the possible values of the 1's digit of d:

1's digit of b	0	1	4	5	6	9
Possibilities for 1's digit of d	0	1, 9	2, 8	5	4, 6	3, 7

3. Where there is a choice of two values for the 1's digit of d, we choose as follows: if $a < c(c+1)$, choose the smaller number; if $a \geq c(c+1)$, choose the larger number.
4. With c and d found, the desired square root is $10c + d$.

Justification

Steps 1 and 2 are clearly true. Only Step 3 needs to be justified, namely: "if $a < c(c+1)$, choose the smaller number; if $a \geq c(c+1)$, choose the larger number." The justification follows from the steps listed below:

(i) If $d \leq 4$, then $a < c(c+1)$. For example, take the case $d = 4$:

$$\begin{aligned}(10c+4)^2 &= 100c^2 + 80c + 16 = 100c^2 + 100c - (20c - 16)\\ &= 100c(c+1) - (20c - 16)\\ &< 100c(c+1), \quad \text{since } c \geq 1.\end{aligned}$$

Hence $N < 100c(c+1)$, implying that $a < c(c+1)$. Similar reasoning takes care of the cases $d = 3, 2, 1$.

(ii) If $d = 5$, then $a = c(c+1)$. This one is easy:

$$(10c+5)^2 = 100c^2 + 100c + 25 = 100c(c+1) + 25,$$

hence $a = c(c+1)$ and $b = 25$.

(iii) If $d \geq 6$, then $a > c(c+1)$. For example, take the case $d = 6$:

$$\begin{aligned}(10c+6)^2 &= 100c^2 + 120c + 36 = 100c^2 + 100c + (20c+36) \\ &= 100c(c+1) + (20c+36) \\ &> 100c(c+1).\end{aligned}$$

Hence $N > 100c(c+1)$, implying that $a > c(c+1)$. Similar reasoning takes care of the cases $d = 7, 8, 9$.

This justifies the stated algorithm.

A side exploration

It is interesting to ask in what cases it happens that $a = c(c+1)$. We have seen that it cannot happen if $d \leq 4$, and it does happen if $d = 5$. What if $d \geq 6$? Let's look at the different cases separately.

- If $d = 6$, then taking forward the working shown above, we see that

$$a = c(c+1) \iff 20c + 36 < 100 \iff 20c < 64 \iff c \leq 3.$$

  So if $d = 6$, then $a = c(c+1)$ provided that $c = 1, 2, 3$. It is easy to verify these claims: $16^2 = 256$, $26^2 = 676$, $36^2 = 1296$. Observe that in each case, we have $a = c(c+1)$.

- If $d = 7$, then we have:

$$(10c+7)^2 = 100c^2 + 140c + 49 = 100c(c+1) + (40c+49).$$

  Hence:

$$a = c(c+1) \iff 40c + 49 < 100 \iff 40c < 51 \iff c \leq 1.$$

  So if $d = 7$, then $a = c(c+1)$ provided that $c = 1$. It is easy to verify this claim: $17^2 = 289$. Observe that we have $a = c(c+1)$.

- If $d = 8$ or 9, then we find by working through the inequalities that there are no cases when $a = c(c+1)$.

In conclusion, we say that the equality $a = c(c+1)$ holds in precisely the following cases: $d = 5$ (i.e., all two-digit numbers whose 1's digit is 5), and the numbers 16, 17, 26 and 36.

References

1. A2Y Academy For Excellence, Finding the Square Root of a number which is a perfect square, http://a2yacademy.com/2018/04/09/finding-the-square-root-of-a-number-which-is-a-perfect-square/

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