PPT EXPLORATIONS

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A classroom observation. A colleague[a] shared an observation that had come up during an exploratory class he was taking at the middle school level. The topic being discussed was ***Pythagorean Triples***, i.e., triples (a, b, c) of positive integers satisfying the relation $a^2 + b^2 = c^2$. If the three integers also happen to be relatively prime to each other, i.e., share no common factors exceeding 1, then the triple is called a 'Primitive Pythagorean Triple' (PPT for short). One of the students pointed out that the most well-known such PPT, namely $(3, 4, 5)$, has the following property:

$$3 + \left(\frac{1}{2} \times 4\right) = 5.$$

Provoked by this observation, we naturally wondered about the existence of Pythagorean triples (a, b, c) which satisfy the relation

$$a + \frac{b}{2} = c.$$

It turns out that the Pythagorean triples that satisfy this relation are all multiples of $(3, 4, 5)$, i.e., $(3, 4, 5)$, $(6, 8, 10)$, $(9, 12, 15)$, ... This means that the *only* PPT which satisfies the stated relation is $(3, 4, 5)$ itself.

It is easy to prove this statement. Suppose that a, b, c are positive integers such that

$$a^2 + b^2 = c^2,$$

$$a + \frac{b}{2} = c.$$

[a]Vinayak Sharma, fellow mathematics teacher in Sahyadri School KFI. Thanks, Vinayak!

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Substituting for c from the second relation in the first one, we get:

$$a^2 + b^2 = a^2 + ab + \frac{b^2}{4},$$

$$\therefore\ 3b^2 = 4ab, \quad \therefore\ 3b = 4a,$$

$$\therefore\ a : b = 3 : 4,$$

implying that $a : b : c = 3 : 4 : 5$.

If a, b, c are to be coprime, then it must be that $(a, b, c) = (3, 4, 5)$. Hence $(3, 4, 5)$ is the only such PPT.

Extending the result. It is easy to extend the result. Let t be any given positive rational number. Suppose that the Pythagorean triple (a, b, c) satisfies the following relation:

$$c = a + tb.$$

We now pose the following question: *What are all the Pythagorean triples for which this relation holds? In particular, what are all the PPTs for which this relation holds?*

The analysis proceeds along the same lines as earlier. Suppose that a, b, c are positive integers such that

$$a^2 + b^2 = c^2,$$

$$a + tb = c.$$

We note in passing that $t > 0$ (because $c > a$) and also that $t < 1$ (because $c < a + b$; this is simply the triangle inequality). For the case considered above, we had $t = 1/2$.

Substituting for c from the second relation in the first one, we get:

$$a^2 + b^2 = a^2 + 2tab + t^2 b^2,$$

$$\therefore\ \left(1 - t^2\right) b^2 = 2tab, \quad \therefore\ \left(1 - t^2\right) b = 2ta,$$

$$\therefore\ a : b = 1 - t^2 : 2t,$$

implying that $a : b : c = 1 - t^2 : 2t : 1 + t^2$.

Now let $t = m/n$ where m and n are coprime integers, $0 < m < n$. Substituting this in the above finding, we get:

$$a : b : c = \frac{n^2 - m^2}{n^2} : \frac{2m}{n} : \frac{n^2 + m^2}{n^2},$$

i.e.,

$$a : b : c = n^2 - m^2 : 2mn : n^2 + m^2.$$

Since we want a, b, c to be relatively prime to each other, we deduce the following:

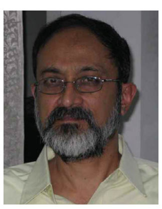
- If m and n have opposite parity (i.e., one of them is even and the other is odd), the greatest common divisor of the quantities $n^2 - m^2$, $2mn$, $n^2 + m^2$ is 1 (since m and n are coprime), hence

$$a = n^2 - m^2, \quad b = 2mn, \quad c = n^2 + m^2.$$

- If m and n have the same parity (this means that m and n are both odd, as we have already stated that they must be coprime), the greatest common divisor of the quantities $n^2 - m^2$, $2mn$, $n^2 + m^2$ is 2, hence

$$a = \frac{n^2 - m^2}{2}, \quad b = mn, \quad c = \frac{n^2 + m^2}{2}.$$

It is interesting that we have obtained the well-known parametrisation of PPTs in a way which is very different from the usual. All this from the simple observation that the entries of the PPT $(3, 4, 5)$ satisfy the relation $3 + (\frac{1}{2} \times 4) = 5$ ….



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