Low Floor High Ceiling Tasks

Squaring the Dots
Think into the Box

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SNEHA TITUS

A square dot sheet has equally spaced dots aligned vertically and horizontally. Many interesting investigations can be devised with this simple learning material. We began with a question that had been posted in the Thinking Skills PullOut (November 2015): *Can you draw a square with just 1 dot inside?* The resulting investigation has branched out in multiple directions; we will explore a few and leave you to try the rest. As usual, we have designed this investigation in the Low Floor High Ceiling style in which we welcome explorers to step easily into the low-floored classroom with fairly easy questions. As the investigations proceed, the questions get more challenging and the high ceiling is designed to keep even the most able students thinking, conjecturing, proving and, in short, being mathematicians.

Task 1: Squares with sides along vertical and horizontal axes.

Note: Squares are always drawn with dots at the vertices.

- How many dots are enclosed within a 1×1 square?
- As the size of the square increases, is there a pattern in the number of dots inside?
- Is it possible to generalise to the number of dots inside an $n \times n$ square?
- Is there more than one way of summing the dots inside the squares?
- Do all these ways of summing the dots give the same formula for the number of dots inside an $n \times n$ square?

Keywords: Dot sheets, squares, counting, slopes, pattern, generalising, algebra

Task 2: Squares with sides inclined to the vertical.

Note: Squares are always drawn with dots at the vertices.

- Draw the smallest square with its diagonals along vertical and horizontal axes. What angle are its sides inclined at?
- If the distance between two adjacent horizontal (or vertical) dots is 1 unit, what is the length of the side of this square?
- Is it possible to get a square with the sides inclined at other angles to the vertical?
- If the distance between two adjacent horizontal (or vertical) dots is 1 unit, find the length of the side of this new square.
- Can you find a general way of generating more and more such squares? Try to find the smallest square each time.
- Can you find a general representation for the sides of the squares thus generated?

Task 3: Counting the dots inside squares with sides inclined to the vertical.

- For each type of square that you have drawn, draw bigger and bigger squares and count the number of dots inside. Make a table for each type of square.
- Can you find a pattern to sum the dots inside?
- Can you find a pattern connecting the number of dots inside to the squares of increasing inclination?

Teacher Notes:

We have simply given **our** final outcomes - an exploration usually starts in an open-ended manner and it is important for students not to be constrained at this time. As they start observing, they will notice patterns and at this time, gentle facilitation will nudge them along the path to more systematic documentation. Do remember that there can be other discoveries that they make and other paths that they may want to follow as their interest deepens.

Task 1: The figures 1.1 and 1.2 and the following table show the results of our investigations.

Side length	Formula 1 for adding dots	Formula 2 for adding dots	Formula 3 for adding dots	No of dots inside
1				0
2				1
3	2 + 2	2 × 2	1 + 2 + 1	4
4	3 + 3 + 3	3 × 3	1 + 2 + 3 + 2 + 1	9
5	4 + 4 + 4 + 4	4 × 4	1 + 2 + 3 + 4 + 3 + 2 + 1	16
n	$(n-1)$ added $(n-1)$ times	$(n-1) \times (n-1)$	$2\,(1 + 2 + \ldots(n-2)) + (n-1)$ $= (n-2)(n-1) + (n-1)$	$(n-1)^2$

Table 1

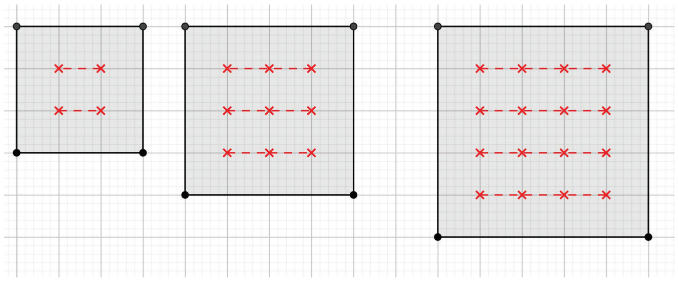

Figure 1.1

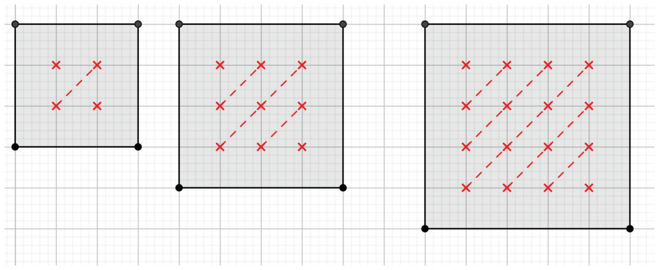

Figure 1.2

Task 2:

Squares with sides at increasing tilts. See Figure 2. Students who are not used to the idea of slope may find it difficult to go beyond the first tilt, but do give them time to explore. The idea of slope and the relationship between slopes of perpendicular lines will emerge very naturally- even without explicit statements.

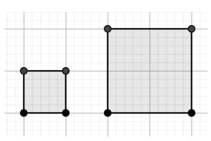

Figure 2.1

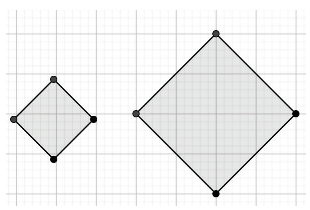

Figure 2.2

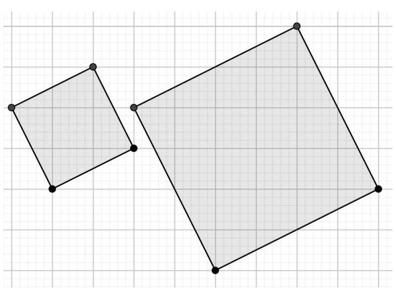

Figure 2.3

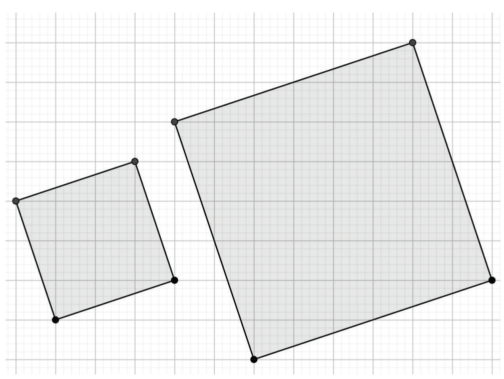

Figure 2.4

Relationship between tilt and length of the side of the square

Figure	Side of square	Tilt in words
2.1	1 unit	Vertical and horizontal sides
2.2	$\sqrt{2}$ units	From the left dot go right 1 and up 1 to get to the dot on the right. Reverse this to go to the dot below.
2.3	$\sqrt{5}$ units	From the left dot, go right 2 and up 1 to get to the dot on the right. Reverse this to go to the dot below.
2.4	$\sqrt{10}$ units	From the left dot, go right 3 and up 1 to get to the dot on the right. Reverse this to go to the dot below.

Table 2

Task 3:

Figure 3 shows the summing patterns obtained for squares with sides increasing in multiples of $\sqrt{2}$. Students can experiment with colour and see newer patterns emerging, which we may not have reported here.

Please note that the grid size had to be adjusted due to constraints of space. Consequently, some of the vertices of the squares may have moved off the dots. However the initial condition of the investigation was preserved, i.e., the vertices of the squares are on the dots.

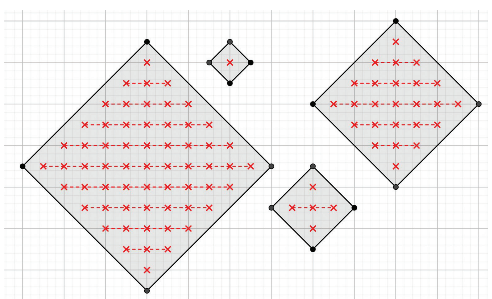

Figure 3.1

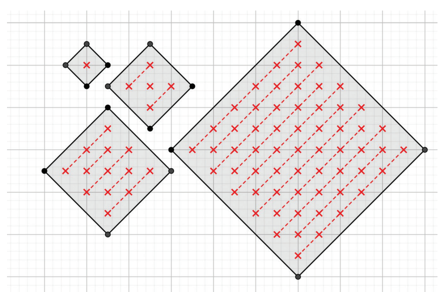

Figure 3.2

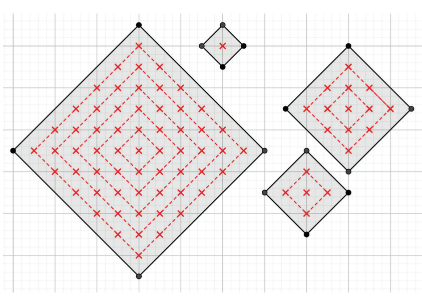

Figure 3.3

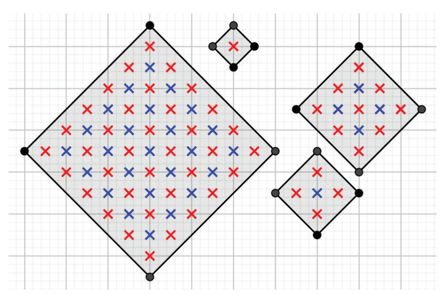

Figure 3.4

Side length	Formula 1 for adding dots	Formula 2 for adding dots	Formula 3 for adding dots	No of dots inside
$\sqrt{2}$	1	1	1	1
$2\sqrt{2}$	$1 + 3 + 1$	$2 + 1 + 2$	$1 + (2 \times 2)$	5
$3\sqrt{2}$	$= 1 + 3 + 5 + 3 + 1$ $= 2(1 + 3) + 5$	$3 + 2 + 3 + 2 + 3$	$1 + (2 \times 2) + (2 \times 3) + (2 \times 1)$ $= 1 + 4 + 8$	13
$6\sqrt{2}$	$= 1 + 3 + 5 + 7 + 9 + 11 + 9 + 7 + 5 + 3 + 1$ $= 2(1 + 3 + 5 + 7 + 9) + 11$	$6 + 5 + 6 + 5 + 6 + 5 + 6 + 5 + 6 + 5 + 6$	$= 1 + (2 \times 2) + (2 \times 3 + 2 \times 1) + (2 \times 4 + 2 \times 2) + (2 \times 5 + 2 \times 3) + (2 \times 6 + 2 \times 4)$ $= 1 + 4 + 8 + 12 + 16 + 20$	61
$n\sqrt{2}$	$= 2(1 + 3 + 5 + \ldots.. (2n - 3)) + 2n - 1$ $= 2(n - 1)2 + 2n - 1$ $= 2n^2 - 2n + 1$	$(n) \times (n) + (n - 1)(n - 1)$ $= 2n^2 - 2n + 1$	$= 1 + (2 \times 2) + (2 \times 3 + 2 \times 1) + \ldots\ldots\ldots (2n + 2(n - 2))$ $= 1 + 4 + 8 + \ldots.. + 4(n - 1)$ $= 1 + 2(n - 1)(n)$ $= 2n^2 - 2n + 1$	$2n^2 - 2n + 1$ $= n^2 + (n - 1)^2$

Table 3

Similarly, other patterns for summation can emerge and so do opportunities for beautiful pictures as students begin to notice sets of patterns. See Figure 4 and Table 4.

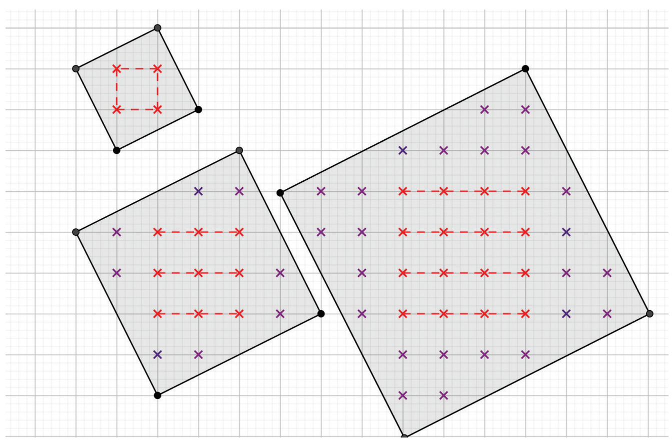

Figure 4

Side length	Formula 1 for adding dots	Formula 2 for adding dots	Formula 3 for adding dots	No of dots inside
$\sqrt{5}$	2×2	$2 + 2$		4
$2\sqrt{5}$	$3 \times 3 + 4 \times 2$	$2 + 4 + 5 + 4 + 2$		17
$3\sqrt{5}$	$= 4 \times 4 + 4 \times 4 + 4 \times 2$	$2 + 4 + 4 \times 7 + 4 + 2$		40

Table 4

Try and find a formula 3 and general representations for each formula.

Mathematisation has never been so much fun and we hope that you have enjoyed the possible avenues of exploration that we have opened up.

Another way to approach this exploration is to take the original question in the Thinking Skills PullOut (http://teachersofindia.org/en/ebook/thinking-skills-pullout), look at the number of dots enclosed by the squares and ask if it is possible to enclose any number of dots with a square.

Clearly 1, 4, 9, 16…. n^2 can be enclosed with squares.

The 45° tilted ones include 1, 5, 13, 25... dots – again a very clear pattern of $n^2 + (n-1)^2$.

Up to 20, the numbers for which it was possible is either $4k$ or $4k + 1$. This can be explained to an extent if you revisit the squares in Figures 1.1, 1.2, 3.1-3.4 and 4. The squares fall in two groups: (i) with a dot at the centre e.g. the squares with 45° tilt and (ii) with no dot at the centre e.g. the biggest square in Figure 4. If you leave the dot at the centre, when it is there, you can split the remaining dots equally in four quadrants. [See Figure 6]. (Clearly, this is related to the four-fold rotational symmetry of the square about its central point.) So if each quadrant has k dots, then squares in (i) have $4k + 1$ dots in them while those in (ii) have $4k$ dots inside.

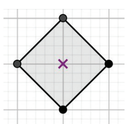

Figure 6.1

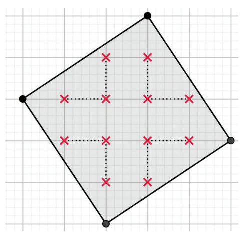

Figure 6.2

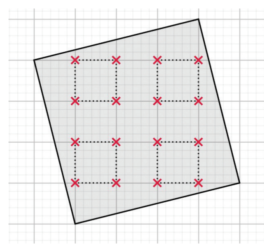

Figure 6.3

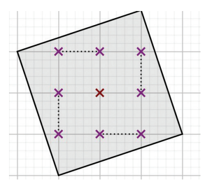

Figure 6.4

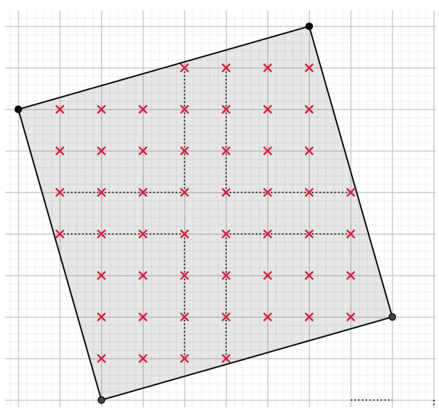

Figure 6.5

Is it possible to 'square' $4k$ or $4k + 1$ dots for each whole number k? Not sure... 20, 21, i.e., $k = 5$ appear to be impossible. Maybe you will find a square that has exactly 20 or 21 dots inside it!

We hope that we have left you with food for thought!

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