PROPERTIES OF MULTIPLICATION

Visual justification for fractions (and whole numbers)

After looking at visual justifications for properties of addition for whole numbers and fractions, this poster considers the properties of multiplication for the same number sets.

The model: A unit square represents the whole, ¼ is represented by slicing a square into 4 equal vertical strips and shading 1 of them, and 5/3 by slicing two squares in 3 strips each and shading 5 strips (Figure 0).

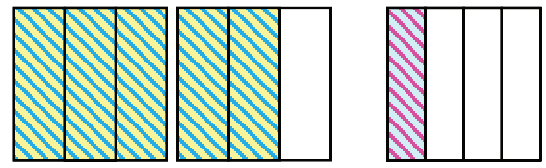


Figure 0

The commutative and distributive properties are easy extensions of the whole numbers case considering the array model of multiplication. In these, the product is represented by the rectangular area whose vertical side represents the 1st multiplicand and the horizontal side represents the 2nd one. So $\frac{2}{3} \times \frac{4}{7}$ is shown by Figure 1 below where the shaded area is 2×4 out of 3×7 parts i.e. $\frac{2\times4}{3\times7}$. Figure 4 illustrates $\frac{4}{3} \times \frac{2}{7}$ where the shaded area is 4×2 parts of the unit square or 3×7 i.e. $\frac{4\times2}{3\times7}$. Note that since $\frac{4}{3}$ is an improper fraction shown along the vertical length, it spills over the unit square (shown with a thick border).

Commutative Property of Multiplication

There are 3 possibilities:

1. Proper × proper
2. Improper × proper (and ∴ proper × improper)
3. Improper × improper

We show one example of 1 and 2 each and 3 can be done in a similar way.

Proper × proper:

$\frac{2}{3} \times \frac{4}{7}$

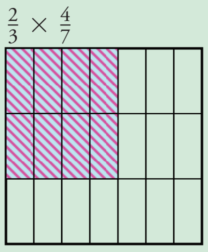


Figure 1

$\frac{4}{7} \times \frac{2}{3}$

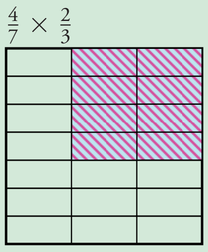


Figure 2

Figure 1 is rotated by 90° to get Figure 2. In Figure 1, the shaded area is 2×4 parts out of 3×7 parts i.e. $\frac{2\times4}{3\times7}$ whereas in Figure 2, it is 4×2 parts out of 7×3 parts i.e. $\frac{4\times2}{7\times3}$.

Note that rotation is a rigid transformation. So the lengths and hence the areas remain unchanged. So $\frac{2}{3} \times \frac{4}{7} = \frac{4}{7} \times \frac{2}{3}$. Therefore this can be generalized for any two proper fractions.

Proper × improper:

$\frac{2}{7} \times \frac{4}{3}$

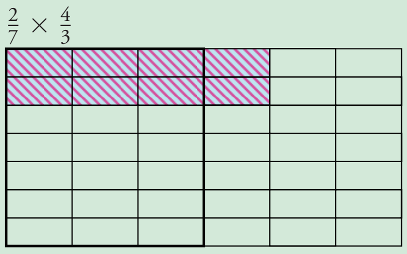


Figure 3

Note:

Figure 3 is rotated by 90° to get Figure 4 as in the previous case. So the areas remain unchanged i.e. $\frac{2}{7} \times \frac{4}{3} = \frac{4}{3} \times \frac{2}{7}$.

In Figure 3, the 2nd multiplicand $\frac{4}{3}$ is an improper fraction. So the shaded area spills over the unit square along the horizontal dimension. In Figure 4, the 1st multiplicand is an improper fraction making the shaded area spill over the unit square along the vertical dimension.

$\frac{4}{3} \times \frac{2}{7}$

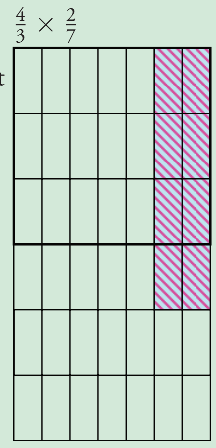


Figure 4

This model can be used to represent whole number × fraction since any improper fraction is a sum of a natural number and a proper fraction. Multiplications involving 0 are by definition 0. So the commutative property is vacuously true when zero is involved.

Distributive Property of Multiplication

There are 6 possibilities:

1. Proper × the sum
   a. Proper × (proper + proper)
   b. Proper × (proper + improper)
   c. Proper × (improper + improper)
2. Improper × the sum
   a. Improper × (proper + proper)
   b. Improper × (proper + improper)
   c. Improper × (improper + improper)

Note that proper + improper = improper + proper by commutativity of addition, which has been discussed in the previous article. We show an example of 1a and 2b each. The reader can explore the remaining.

Proper × (proper + proper)

$\frac{3}{4} \times \left(\frac{4}{7} + \frac{2}{3}\right) = \frac{3}{4} \times \frac{4}{7} + \frac{3}{4} \times \frac{2}{3}$

The total horizontal length of both shaded areas represents the sum $\frac{4}{7} + \frac{2}{3}$ while the vertical length is $\frac{3}{4}$. So the total shaded area is $\frac{3}{4} \times \left(\frac{4}{7} + \frac{2}{3}\right)$. On the other hand, pink area represents $\frac{3}{4} \times \frac{4}{7}$ while the green region depicts $\frac{3}{4} \times \frac{2}{3}$. So the total shaded area is $\frac{3}{4} \times \frac{4}{7} + \frac{3}{4} \times \frac{2}{3}$. Therefore $\frac{3}{4} \times \left(\frac{4}{7} + \frac{2}{3}\right) = \frac{3}{4} \times \frac{4}{7} + \frac{3}{4} \times \frac{2}{3}$

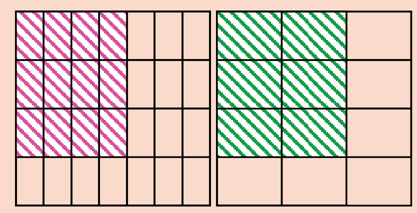


Figure 11

Improper × (proper + improper):

$\frac{5}{3} \times \left(\frac{2}{7} + \frac{7}{4}\right) = \frac{5}{3} \times \frac{2}{7} + \frac{5}{3} \times \frac{7}{4}$

Figure 12 can be understood along the same lines as Figure 11. Note that the shaded area spills over the unit square whenever an improper fraction is involved and along that dimension.

This model works for visually justifying the distributive property of multiplication for any combination of three fractions (and natural numbers including zero).

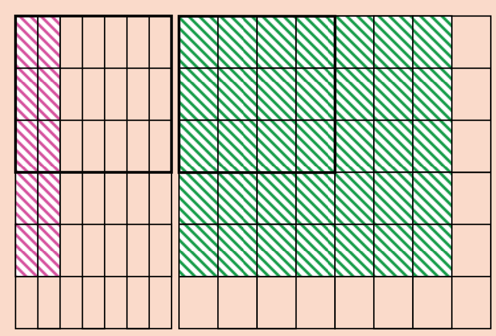


Figure 12

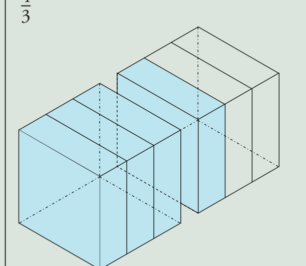
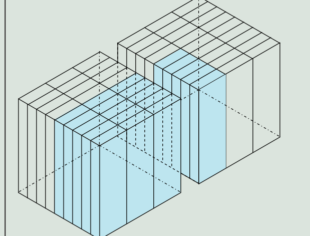
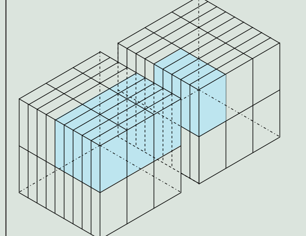
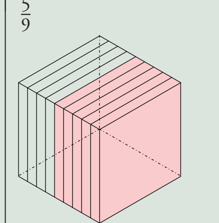
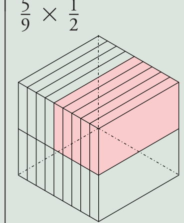
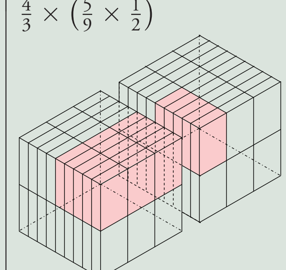
Associative Property of Multiplication

Associative property of multiplication involves the product of three fractions. So we use volume and 3D i.e. the unit cube as the whole and represent the three fractions along the y, x and z axes respectively.

There are 8 possibilities:

1. All three proper
2. Two proper and one improper
   a. Proper × proper × improper
   b. Proper × improper × proper
   c. Improper × proper × proper
3. One proper and two improper
   a. Proper × improper × improper
   b. Improper × proper × improper
   c. Improper × improper × proper
4. All three improper

We show an example of 2c and the rest can be done in a similar manner. Given the 3D representation we show it step-wise. We consider the example $\left(\frac{4}{3} \times \frac{5}{9}\right) \times \frac{1}{2} = \frac{4}{3} \times \left(\frac{5}{9} \times \frac{1}{2}\right)$

	Step 1	Step 2	Step 3
Left hand side: $\left(\frac{4}{3} \times \frac{5}{9}\right) \times \frac{1}{2}$	$\frac{4}{3}$  Figure 5	$\frac{4}{3} \times \frac{5}{9}$  Figure 6	$\left(\frac{4}{3} \times \frac{5}{9}\right) \times \frac{1}{2}$  Figure 7
Right hand side: $\frac{4}{3} \times \left(\frac{5}{9} \times \frac{1}{2}\right)$	$\frac{5}{9}$  Figure 8	$\frac{5}{9} \times \frac{1}{2}$  Figure 9	$\frac{4}{3} \times \left(\frac{5}{9} \times \frac{1}{2}\right)$  Figure 10

For visual clarity, we have kept a gap between the two unit cubes. But that need not be there.
Note that the end volume is the same for both and that this can be generalized for any three fractions (and natural numbers).

This poster, created by SWATI SIRCAR, provides visual justifications which are not the same as proofs. It is based on the PullOuts by Padmapriya Shirali – Multiplication (Vol. 3, Issue 1) and Fraction (Vol. 1, Issue 1), and Visual Method for Fraction Multiplication by Shailaja D Sharma (Vol. 7, Issue 2)