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Arc - centre Theorem

HAREN SATHVIK

Student Haren Sathvik uses mathematics as a power tool to win an argument and prove his point. Exactly what the Position Paper of the National Focus Group on Mathematics says: Children use abstractions to perceive relationships, to see structure, to reason about things, to argue the truth or falsity of statements. Logical thinking is a great gift mathematics can offer us, and inculcating such habits of thought and communication in children is a principal goal of teaching mathematics. Read on to find out more.

In this short note, we discuss a theorem (arc-centre theorem) which states that an arc can have only one centre of curvature.

Centre of curvature: The centre of a circle of which the given arc is a part.

Introduction

When my classmates and I were preparing for the Regional Mathematical Olympiad, we were given the following problem:

A circle of radius 2 is centred at O. Suppose square OABC has side length 1; sides AB and CB are extended past B to meet the circle at D and E respectively (refer Figure 1). What is the area of the shaded region in the figure which is bounded by BD, BE and the minor arc connecting D and E?

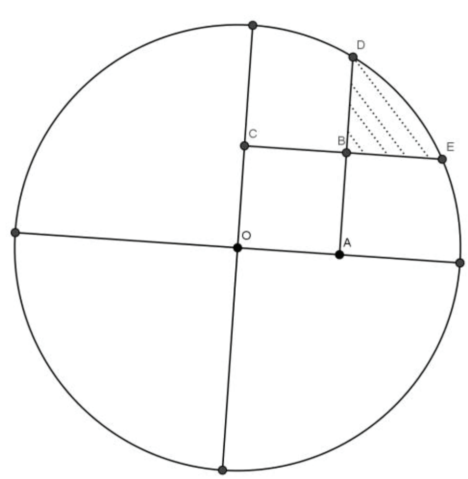

Figure 1.

Keywords: Centre of curvature, arc

I solved the problem using trigonometry. My friend Mahesh came up with a different argument. His approach was to construct another circle with centre B and then argue that arc DE must be its quarter circumference (since angle DBE $= 90°$). I felt this was wrong, as you cannot have two different circles (with two different centres) sharing an arc (i.e., arc DE). When we asked for help from our math mentor, he challenged us to prove the statement. I came up with a proof which I give in this note. That is, I prove the "arc centre theorem" – the statement that "A given arc has a unique centre of curvature."

Before proving the theorem, let us show how to construct a centre for a given arc using straightedge and compass.

Construction. Draw any two non-parallel chords with endpoints on the arc. For each of them, draw a second chord parallel to the first, also with endpoints on the arc. Construct the midpoint of each chord. For each pair of parallel chords, draw the line through their midpoints. The two lines then intersect at the centre of the circle [1].

Now let us prove our main theorem.

Theorem (Arc-centre theorem)

An arc of a circle can have only one centre of curvature.

Proof. Consider arc $\overline{ABC}$ where A and C are the endpoints and B is the midpoint of AC. Let L be a line through B, perpendicular to the tangent to arc $\overline{ABC}$ at B; that is, L is the normal to arc $\overline{ABC}$ at B (refer Figure 2). Suppose that it has more than one centre of curvature (in contradiction to the statement of the theorem). Let the two centres be O_1 and O_2 (refer Figure 3).

Assume $O_1O_2 = h$. Then we have:

$$AO_1 = BO_1 = CO_1 = r_1$$
$$AO_2 = BO_2 = CO_2 = r_2$$

Now we use the triangle inequality which states that in a triangle, the sum of any two sides is greater than the third side. So from triangle AO_1O_2,

$$AO_1 + O_1O_2 > AO_2, \text{ i.e., } r_1 + h > r_2 \qquad (1)$$

But since B, O_1, O_2 lie on a straight line,

$$BO_1 + O_1O_2 = BO_2, \text{ i.e., } r_1 + h = r_2 \qquad (2)$$

It is clear that (2) contradicts (1). Hence the initial supposition must be wrong. Therefore, we can say two different centres of curvatures do not exist for a single arc. Hence proved.

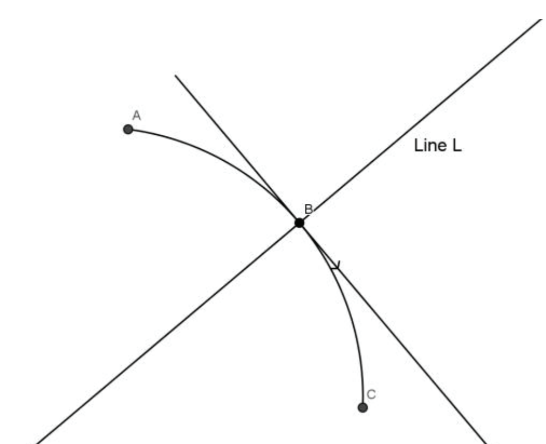

Figure 2.

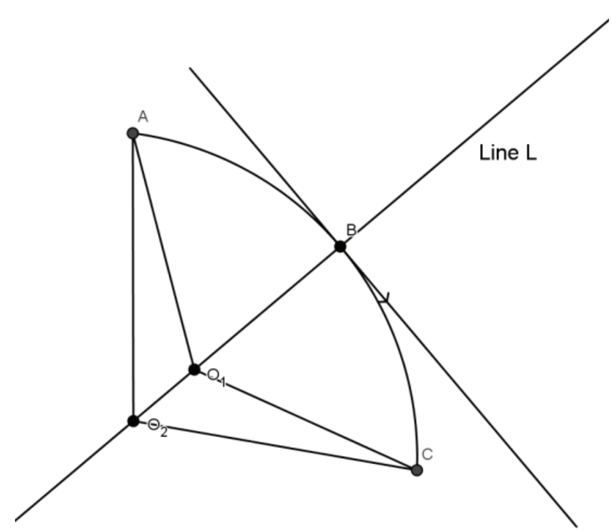

Figure 3.

Acknowledgement

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Reference

1. https://math.stackexchange.com/questions/1113276/how-to-find-centre-of-a-circle-from-only-an-arbitary-arc-of-that-circle

HAREN SATHVIK is a grade 9 student of Narayana CO Sindhu Bhavan, Bangalore. He has qualified in multiple math Olympiads, doing well both at the state level and the national level. He has also done well in the state level Abacus competition and the mathematical Olympiad (SOF). He has a keen interest in Number Theory, Algebra and Combinatorics. His hobbies are listening to songs, solving Olympiad problems, reading books and playing cricket. He wants to become a scientist and wants to solve unsolved problems in number theory. He is passionate about lightning as a source of energy. He thanks his parents and teachers for what he has learned. He may be contacted at haren.sathvik1729@gmail.com.