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Solutions to Practice Problems in Functional Equations – Part I

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Problem 1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = (x - y)(f(x) + f(y))$$

for all $x, y \in \mathbb{R}$.

Solution 1. Let $P(x, y)$ be the stated property. Note that it implies that f is odd (this may be seen by interchanging x and y).

$$\begin{aligned} P(0,0): &\quad f(0) = 0, \\ P(x,0): &\quad f(x^2) = xf(x), \\ P(0,x): &\quad f(-x^2) = -xf(x), \\ P(x,-y): &\quad f(x^2 - y^2) = (x+y)(f(x) - f(y)). \end{aligned}$$

Comparing the last line with $P(x, y)$ gives

$$(x-y)(f(x)+f(y)) = (x+y)(f(x)-f(y)) \implies yf(x) = xf(y).$$

Hence for $x, y \neq 0$,

$$\frac{f(x)}{x} = \frac{f(y)}{y}.$$

Hence $\frac{f(x)}{x} = c$ for some constant $c \in \mathbb{R}$ and for all $x \neq 0$. Since $f(0) = 0$, $f(x) = cx$ for all x satisfies the original problem statement. Hence $f(x) = cx$ for all $x \in \mathbb{R}$.

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Problem 2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(4x) - f(3x) = 2x$$

for all $x \in \mathbb{R}$.

Solution 2. The problem may be restated as

$$f(x) - f\left(\frac{3x}{4}\right) = \frac{x}{2}, \quad \text{i.e., } f(x) = f\left(\frac{3x}{4}\right) + \frac{x}{2}.$$

Iterating this relation, we get

$$\begin{aligned} f(x) &= f\left(\frac{3x}{4}\right) + \frac{x}{2} \\ &= f\left(\frac{9x}{16}\right) + \frac{x}{2} + \frac{3x}{8} \\ &= f\left(\frac{27x}{64}\right) + \frac{x}{2} + \frac{3x}{8} + \frac{9x}{32} + \dots. \end{aligned}$$

Let $a = \lim_{x\to 0^+} f(x)$. Thus we have for $x > 0$,

$$\begin{aligned} f(x) &= a + \frac{x}{2} + \frac{3x}{8} + \dots \\ &= a + \frac{x}{2}\left(1 + \frac{3}{4} + \frac{9}{16} + \dots\right) = a + 2x. \end{aligned}$$

Let $b = \lim_{x\to 0^-} f(x)$. By reasoning in the same way, we get $f(x) = b + 2x$ for $x < 0$.

Hence $f(x) = 2x + a$ for all $x > 0$, $f(x) = 2x + b$ for all $x < 0$ and $f(0) = c$ for some constants a, b, c. (Note that if f is to be continuous, then we must have $a = b = c$.)

Problem 3. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y)$$

for all $x, y, z \in \mathbb{R}$.

Solution 3. Let $P(x, y, z)$ be the stated property.

$$\begin{aligned} P(0,0,0):&\quad f(0) = 0; \\ P(x,0,0):&\quad f(x^2) = xf(x); \\ P(0,y,z):&\quad f(yf(z)) = zf(y), \quad \therefore\ f(f(z)) = zf(1). \end{aligned}$$

Suppose $f(1) = 0$.

$$P(x, y, 1):\quad xf(x) = f(x^2) = xf(x) + f(y), \quad \therefore\ f(y) = 0 \text{ for all } y.$$

Next, suppose $f(1) \neq 0$. As $f(f(z)) = zf(1)$, f is injective. (For, if $f(a) = f(b)$, then $f(f(a)) = f(f(b))$, hence $af(1) = bf(1)$, hence $a = b$.) We also had $f(yf(z)) = zf(y)$. Plugging $y = z$, we get

$$f(yf(y)) = yf(y) = f(y^2), \quad \therefore\ yf(y) = y^2,$$

hence $f(y) = y$ for all $y \neq 0$. As $f(0) = 0$, $f(y) = y$ for all y.

Hence the solutions are $f(x) = x$ for all $x \in \mathbb{R}$ and $f(x) = 0$ for all $x \in \mathbb{R}$.

Problem 4. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\left(\frac{f(x)}{y}\right) = yf(y) \cdot f(f(x))$$

for all $x, y \in \mathbb{R}^+$.

Solution 4. Let $P(x, y)$ be the stated property. Since 0 is not an element of the domain, we first try to find $f(1)$.

$$P(1,1): \quad f(f(1)) = f(1) \cdot f(f(1)), \quad \therefore\ f(1) = 1.$$

Now LHS has a term $f\left(\frac{f(x)}{y}\right)$, we substitute in such a way that LHS becomes 1:

$$P(x, f(x)): \quad 1 = f(1) = f(x) \cdot f(f(x,))^2, \quad \therefore\ f(f(x)) = \frac{1}{\sqrt{f(x)}},$$

$$P(x, f(y)): \quad f\left(\frac{f(x)}{f(y)}\right) = f(y) \cdot f(f(y)) \cdot f(f(x)) = \sqrt{\frac{f(y)}{f(x)}}.$$

We cannot conclude from this that $f(x) = \frac{1}{\sqrt{x}}$ for all x; we first need to show that $\frac{f(x)}{f(y)}$ is surjective. We had

$$\frac{f\left(\frac{f(x)}{y}\right)}{f(y)} = y \cdot f(f(x)).$$

Clearly RHS is surjective (with an isolated y), hence so is LHS. Hence we can substitute $\frac{f(x)}{f(y)}$ as some positive real a. Therefore we have $f(a) = \sqrt{\frac{1}{a}}$.

Hence $f(x) = \frac{1}{\sqrt{x}}$ for all $x \in \mathbb{R}^+$.

Problem 5. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(x + y^2) \cdot f(y \cdot f(x)) = x \cdot y \cdot f(y^2 + f(x))$$

for all $x, y \in \mathbb{R}$.

Solution 5. Let $P(x, y)$ be the stated property. Clearly $f(x) = 0$ is a solution. We wish to explore whether there is a nontrivial solution, so let us assume that there exists some real x such that $f(x) \neq 0$. We now try to find $f(0)$:

$$P(x, 0): \quad x \cdot f(0) = 0 \text{ for all } x, \quad \therefore\ f(0) = 0.$$

Notice that there are no x terms inside a f-term except for $f(x)$. This might help us showing that f is injective. Let $f(a) = f(b)$. Let $y \neq 0$ and $a + y^2 \neq 0$, $b + y^2 \neq 0$. $P(a, y)$ and $P(b, y)$ give:

$$\frac{ayf(y^2 + f(a))}{a + y^2} = f(yf(a)) = f(yf(b)) = \frac{byf(y^2 + f(b))}{b + y^2}.$$

This yields

$$\frac{a}{a + y^2} = \frac{b}{b + y^2}.$$

Therefore $ab + by^2 = ab + ay^2 \Rightarrow a = b \implies f$ is injective.

The left hand side contains a term $(x + y^2)$, let us utilise this term effectively.

$$P(-y^2, y): \quad 0 = -y^3 f(y^2 + f(-y^2)).$$

Choose $y \neq 0$, we get $0 = y^2 + f(-y^2) \implies f(-y^2) = -y^2$. Now y is any real number, so $-y^2$ can assume any negative real value. Therefore $f(x) = x$ for all $x < 0$. Let $x, y > 0$.

$$P(-x, y): \quad (y^2 - x)f(-xy) = -xyf(y^2 - x),$$

$$\therefore (y^2 - x)(-xy) = -xyf(y^2 - x),$$

$$\therefore y^2 - x = f(y^2 - x) \text{ for all } x, y.$$

Therefore $f(x) = x$ for all $x \in \mathbb{R}$.

Hence $f(x) = 0$ for all $x \in \mathbb{R}$ and $f(x) = x$ for all $x \in \mathbb{R}$.

Problem 6. Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$f(3x) - f(x) \leq 8x^2 + 2x, \quad f(2x) - f(x) \geq 3x^2 + x.$$

Solution 6. This is an unique problem since the problem is a functional inequality, which we have not encountered before. How do we proceed now? One of the approaches to solve a FE is to guess the solution. In this case $f(x) = x^2 + x$ is one of the solutions. (Note that for this function, equality holds in both the relations: $f(3x) - f(x) = 8x^2 + 2x$ and $f(2x) - f(x) = 3x^2 + x$.)

Therefore it makes sense to do the following substitution:

$$g(x) = f(x) - (x^2 + x).$$

Now how do the inequalities change?

$$g(3x) - g(x) = f(3x) - 9x^2 - 3x - f(x) + x^2 + x = (f(3x) - f(x)) - 8x^2 - 2x \leq 0$$

and

$$g(2x) - g(x) = f(2x) - 4x^2 - 2x - f(x) + x^2 + x = (f(2x) - f(x)) - 3x^2 - x \geq 0.$$

Hence we have:

$$g(3x) \leq g(x), \quad g(2x) \geq g(x).$$

We also have an additional condition of continuity. Hence:

$$g(x) \leq g\left(\frac{x}{3}\right) \leq g\left(\frac{x}{3 \cdot 3}\right) \leq \cdots \leq g\left(\frac{x}{3 \cdot 3 \cdot \cdots}\right) \leq g(0),$$

and

$$g(x) \geq g\left(\frac{x}{2}\right) \geq g\left(\frac{x}{2 \cdot 2}\right) \geq \cdots \geq g\left(\frac{x}{2 \cdot 2 \cdot \cdots}\right) \geq g(0).$$

Hence $g(x) = g(0)$. Let $g(0) = c$; then we get $g(x) = c$, hence $f(x) = x^2 + x + c$ which certainly does solve the functional inequality.

Hence $f(x) = x^2 + x + c$ for all $x \in \mathbb{R}$ and for some constant c.

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