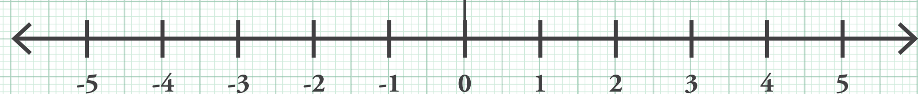
At Right Angles

A Resource for School Mathematics

INTRODUCTION TO ALGEBRA

PADMAPRIYA SHIRALI



Azim Premji University

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

INTRODUCTION

Introduction to school algebra can happen through varied approaches. Some prefer to start with an unknown in an equation, while some prefer to start with a formula and some others may prefer to use a pattern based approach. Does it make a difference which approach one uses? Is one approach better than the others? These questions can be debated. However, each of these approaches relates to different conceptions of algebra.

The unknown in an equation conceives algebra as a study of procedures for solving certain kinds of problems requiring simplification; the formula approach conceives algebra as the study of relationship among quantities which vary. The pattern based approach conceives algebra as generalized arithmetic leading to generalization of known relationships among numbers.

Algebra thus is all of these: generalized arithmetic, a procedure for solving certain problems, and a means of understanding relationships and mathematical structures.

In school algebra, the term 'variable' typically appears first in the form of a letter that represents an unknown in an open sentence or an equation (e.g., $4 + x = 9$), followed by formulas (e.g., $A = L \times B$), as a generalized property (e.g., $a + b = b + a$), later as an identity (e.g., $(a + b)^2 = a^2 + 2ab + b^2$) and as a function (e.g., $y = 3x$). Students learn to use variables to solve various types of problems.

However, does algebraic thinking take place in a child's mind well before he/she encounters a variable? For instance, when a child says 'I have 6 toffees; if there were 4 more I would have 10" or when a child is able to abstract a pattern from numerical relationships, or when a child is able to guess the tenth figure in a pattern of figures, can one say that the child has begun to think algebraically?

The late Shri P. K. Srinivasan had developed an approach to the teaching of algebra titled *'Algebra – a language of patterns and designs'*. I have used it for several years at the Class 6 level and found it to be very useful in making a smooth introduction to algebra, to the idea and usage of concepts such as *variable* and *constant*, to performing operations involving terms and expressions. This approach steadily progresses from studying numerical patterns to line and 2-D designs, finally leading to indices and identities. Over the years, I have adapted this material to meet the needs and interests of the students. However, the basic structure has remained largely the same. I share here the adapted approach.

Patterns, numerical or visual, have an inherent appeal to children and adults alike. It may have to do with the aesthetic feeling present in the human psyche. We are able to recognise and sense patterns in nature, patterns in the movements of the heavenly bodies, patterns in time (seasons) – patterns on a macro-scale as well as on a micro-scale.

Patterns make a very good starting point for the introduction of algebra. They arise easily from the mathematical knowledge that students have already acquired by Class 6 (even and odd numbers, multiplication tables, behaviour of certain numbers, number relationships).

In this pull-out, I focus on the first step of working with patterns as an introductory step to the usage of concepts such as Variable, Constant, Term and Expression and also operations involving these concepts. In the second pull-out I will take up design language and depict the usage of the same concepts and operations. In the third pull-out I will take up indices and identities. Subsequently, approaches to equations will be taken up.

Keywords: Algebra, unknowns, equations, expressions, patterns, activities, manipulatives.

ACTIVITY 1

Objective: To expose students to different kinds of patterns

Pattern recognition is innate to the brain and happens quickly and naturally. However, if students have been taught earlier through a rote and mechanical approach, one may need to reawaken their observation and thinking powers.

Pattern problems in numbers and designs are available in plenty as resources. The teacher will need to make an appropriate graded selection suited to the needs of the upper primary kids.

I have given here a few model problems.

1. What is the pattern here? What goes into the blank space?
   a. 7, ____, 24, 34, 45, 57, 70
   b. 71, 70, 73, 72, 75, _____, _____, _____
2. Find the odd one out. Justify your answer.
   a. 252, 72, 1, 275, 24, 488

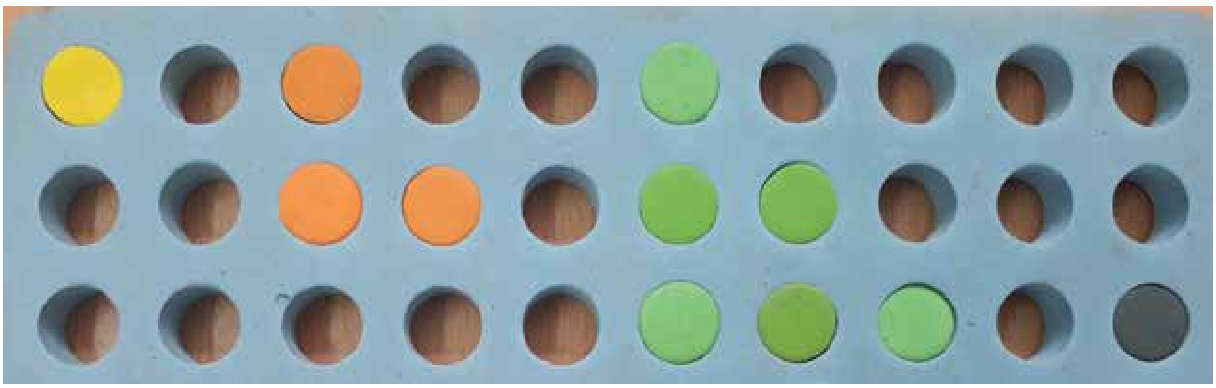


Figure 1

1. Here are the first five *triangular numbers*: 1, 3, 6, 10, 15.
2. Can you see a pattern?
3. Can you predict the next triangular number?
4. What would the tenth triangular number be?

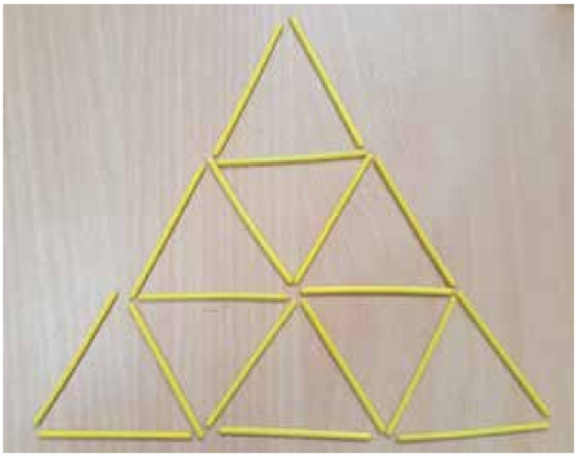


Figure 2

1. I used 3 matches to make 1 small triangle.
2. How many matches do I need to build a second row of triangles under that?
3. How many matches do I need to build a third row of triangles under that?
4. How many matches will I need to make the sixth row?
5. Can you make out how many matches I will need to make the twentieth row?

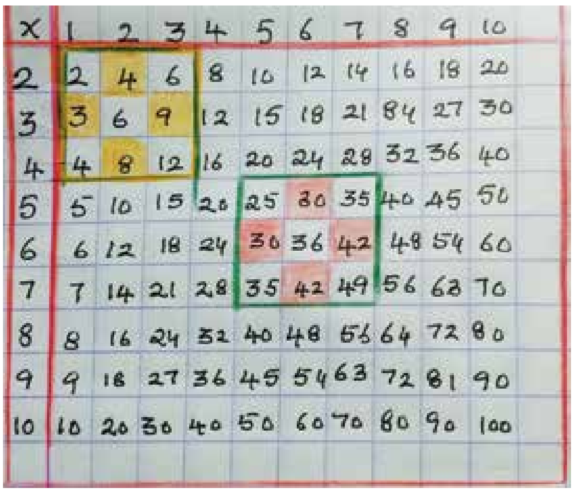
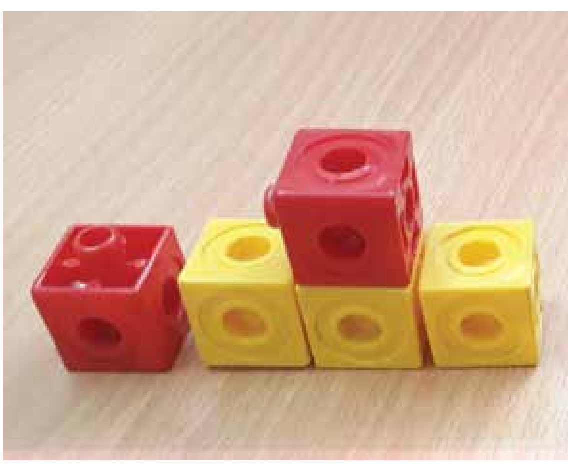


Figure 3

1. Make a box around a set of nine numbers (a 3 × 3 square) in the tables square.
   a. Add the numbers in the shaded squares.
   b. Add the corner numbers.
   c. Multiply the centre number by 4. What happens?
2. Make a box around another set of nine numbers and try this again.



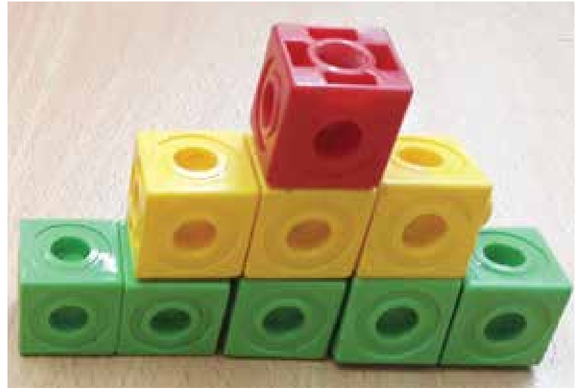


Figure 4

1 block is needed to make an up-and-down staircase, with 1 step up and 1 step down.

4 blocks are needed to make an up-and-down staircase with 2 steps up and 2 steps down.

How many blocks would be needed to build an up-and-down staircase with 5 steps up and 5 steps down?

Explain how you would work out the number of blocks needed to build a staircase with any number of steps.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 5

A hundred square has been printed on both sides of a piece of paper. One square is directly behind the other just like in the pages of a book.

What number lies on the other side of 100? The other side of 58? Of 23? Of 19?

Do you see a pattern?

ACTIVITY 2

Objective: Introduction to pattern language and the usage of a letter for changing numbers.

Introduction to the notion of a *changing number* (variable) and an *unchanging number* (constant) can begin in familiar settings. We introduce the words constant and variable, term and expression a little later.

Students already know about even numbers and about multiples and square numbers. This activity helps them to learn to write pattern language in the context of their prior knowledge of number relationships.

Give students a set of even numbers. For example, 12, 22, 8, 44.

Pose the question, "What are these?" They will notice that they are all even numbers.

What else can be said about them? They are all multiples of 2.

Now the teacher can rewrite all these numbers as multiples of 2.

$$22 = 2 \times 11$$

$$8 = 2 \times 4$$

$$44 = 2 \times 22$$

Now pose the question 'What do you notice about the right hand side?' What is the first number? It is always 2. What is the second number? It is changing each time.

So how can we describe an even number? It is 2 times some number.

Since the second number changes or varies, we represent it using a letter.

An even number can be now written as 2 times 'n' or $2 \times n$. (Mention to the students that we drop the multiplication sign as it looks like the letter 'x'. So $2n$ means '2 times n').

We can take up another example using multiples.

44,11, 220, 121.

What are these? They are all multiples of 11.

They can be written in this way.

$$44 = 11 \times 4$$

$$11 = 11 \times 1$$

$$220 = 11 \times 20$$

$$121 = 11 \times 11$$

What do we see on the right side? The first number is always 11. The second number is changing.

This pattern can be written as $11x$ or $11y$. (Tell the children that any letter can be used to stand for the changing number).

Let us take up a slightly different example where there is no constant factor.

16, 49, 4, 81.

What are these numbers? Square numbers.

They can be written in this way.

$$16 = 4 \times 4$$

$$49 = 7 \times 7$$

$$4 = 2 \times 2$$

$$81 = 9 \times 9$$

What can we say about the numbers on the right hand side? Help the students articulate this. *'The first number is changing. The second number too is changing. But the first number and the second numbers are always the same.'* So, how does one describe such a pattern?

It can be described as 'y' times 'y' or '$y \times y$'or 'yy'. (Note: At this point, we do not write yy as y^2 as we have not yet introduced indices to them.)

Let us take up another type of situation where both the factors are different variables.

Here are some numbers. Can you write them as product of two numbers without using 1 as a factor?

65, 14, 6, 77.

We write them as products:

$$65 = 5 \times 13$$

$$14 = 2 \times 7$$

$$6 = 2 \times 3$$

$$77 = 7 \times 11$$

On the right hand side, what can we say about the first number? The second number? *They are both changing*. The first changing number can be called 'x' and the second changing number can be called 'y'. The pattern here can be described as 'x' times 'y' or '$x \times y$' or 'xy'.

The teacher can ask the students to come up with more such examples of their own.

Students can also work in pairs. Each student can create a pattern using multiples of slightly larger numbers, say, between 10 and 20 and ask another to describe the pattern using pattern language. Or they can do the same with cube numbers.

ACTIVITY 3

Objective: Patterns with two terms and two operations.

Let us look at these numbers.

21, 43, 7, 101.

What are these numbers? They are odd. How do we describe them? Students may take time to respond to that question.

Another question which can help is: 'What is their relationship to even numbers?' They are either 1 more or 1 less than even numbers.

So we write them initially as follows:

$21 = 20 + 1$

$43 = 42 + 1$

$7 = 6 + 1$

$101 = 100 + 1$

At this point, we can describe them as $n + 1$. Is there something further we can do? How did we describe the even numbers earlier? So now we write these numbers as follows:

$21 = 20 + 1 = 2 \times 10 + 1$

$43 = 42 + 1 = 2 \times 21 + 1$

$7 = 6 + 1 = 2 \times 3 + 1$

$101 = 100 + 1 = 2 \times 50 + 1$

Now we describe the pattern as $2n+1$.

Students can be shown that the same numbers, expressed differently, can be described as $2n - 1$.

Note: At this point the teacher can introduce the words variable, constant, term and expression to the students.

Here is another pattern.

49, 69, 19, 89.

All the numbers end with a 9 in the units place. They can be written as follows.

$49 = 10 \times 5 - 1$

$69 = 10 \times 7 - 1$

$19 = 10 \times 2 - 1$

$89 = 10 \times 9 - 1$

The pattern here is $10n - 1$.

Let us look at another pattern which uses place value.

36, 75, 49, 81, 19.

What pattern can one see here? They are not all composite. They are not multiples of any single number. They are all double digit numbers. They can be written as follows:

$36 = 10 \times 3 + 6$

$75 = 10 \times 7 + 5$

$49 = 10 \times 4 + 9$

$81 = 10 \times 8 + 1$

$19 = 10 \times 1 + 9$

This pattern can be described as $10m+n$.

How about this set?

94, 99, 91, 95.

They could be expanded as follows:

$94 = 100 - 6 = 10 \times 10 - 6$

$99 = 100 - 1 = 10 \times 10 - 1$

$91 = 100 - 9 = 10 \times 10 - 9$

$95 = 100 - 5 = 10 \times 10 - 5$

Hence the pattern becomes $10 \times 10 - n$.

Students may also see it as $90 + n$.

Game: Pattern detective

Objective: To detect the pattern created by another.

Materials: Black board or blank paper

This game can be played by the whole class or by small groups of 5 students or even in pairs.

Student I calls out any number between 1 and 10, say 5. Student II performs any two operations on the given number to generate a new number, say 12. This exchange between student I and student II is repeated at least four times. Each time student II performs the same operations in the same order to generate corresponding numbers.

This is how it goes.

Student I	Student II
5	12
3	8
8	18
10	22

What is student II doing with the numbers given by student I?

The pattern needs to be detected by either student I or the group or the class that is watching.

Here student II is doubling the number and adding 2 to the product.

The pattern can be described as $2n+2$.

Note: Initially it is better for the students to use two specified operations, i.e., either '× and +' or '× and –'.

Here is another example of this game between the two students.

Student I	Student II
5	24
3	8
8	63
10	99

What is student II doing with the numbers given by student I?

Here student II is squaring the number and subtracting 1 from the product.

The pattern can be described as '$nn - 1$'.

Here is one more example of this game between two students.

Student I	Student II
1	3
2	7
3	11
4	15

What is student II doing with the numbers given by student I?

I will leave it to you to figure out!

ACTIVITY 4: PATTERNS IN EXPRESSIONS

Objective: To describe given patterns and create patterns for a given expression

To observe addition of like terms with a single variable

$$2 \times 3 + 3 \times 3$$

$$2 \times 5 + 3 \times 5$$

$$2 \times 2 + 3 \times 2$$

$$2 \times 1 + 3 \times 1$$

How do we describe the pattern here?

Let the students state that it is $2a + 3a$.

Now ask the students to work out the sum for each expression and write it as shown.

Ask them to find a pattern in the answers. They will see that they are multiples of 5.

Let them write the answer as multiples of 5.

2 × 3 + 3 × 3	15	5 × 3
2 × 5 + 3 × 5	25	5 × 5
2 × 2 + 3 × 2	10	5 × 2
2 × 1 + 3 × 1	5	5 × 1

How will this pattern be described? It will be 5*a*.

Teacher can point out to the students the fact that 2*a* and 3*a* have summed up to 5*a*.

Now pose the question: "What would 3*x* and 4*x* add up to?" Let the students guess and build patterns to verify their answer.

It is important at this point to show that when unlike terms are added, the answer cannot be 'simplified'.

Provide a number pattern like this:

3 × 3 + 2 × 4	17
3 × 5 + 2 × 2	19
3 × 2 + 2 × 7	20
3 × 1 + 2 × 3	9

How will the pattern on the left hand side be described? It is 3*a* + 2*b*.

Can the students find any pattern in the sums of these numbers?

They can now try to guess the answer for a subtraction situation, e.g., 5*x* – 2*x*, and build a pattern to verify the answer.

ACTIVITY 5: PATTERNS IN EXPRESSIONS

Objective: To describe given patterns and create patterns for a given expression

To observe addition of like terms with more than one variable

How will this pattern be described?

2 × 4 + 4 × 2
3 × 6 + 6 × 3
5 × 2 + 2 × 5
8 × 3 + 3 × 8

It is of the form *ab* + *ba*.

Here again the students can sum them and observe the results.

What is the pattern in the answers? They are all multiples of 2.

Let the students write them initially as multiples of 2 (16 = 2 × 8, etc).

As a second step, they can write the factors of the second number as well (16 = 2 × 2 × 4, etc).

2 × 4 + 4 × 2	2 × 8	16	2 × 2 × 4
3 × 5 + 5 × 3	2 × 15	30	2 × 3 × 5
6 × 7 + 7 × 6	2 × 42	84	2 × 6 × 7
3 × 3 + 3 × 3	2 × 9	18	2 × 3 × 3

What is the pattern of the answers in the final column?

It is 2*ab*.

Again draw the students' attention to the addition of *ab* + *ba* which equals 2*ab*.

Are *ab* and *ba* like terms? Why?

Discuss more examples of 'like' and 'unlike' terms in two variables.

As a practice exercise, students can be asked to set up a number pattern for a given pattern language, using only like terms initially.

Ex. Create number patterns for *xy* + *xy* + *xy*.

What does it become?

Would it be different for *xy* + *yx* + *xy*?

Create number patterns for 5*cd* – 2*cd*.

What does it become?

Let the students also create patterns for addition and subtraction of unlike terms.

Example: Create number patterns for each:

(i) *abc* – *cde* (ii) *ab* + *bc* + *ca*.

ACTIVITY 6: LAWS OF COMMUTATIVITY AND ASSOCIATIVITY

Objective: To establish commutativity and associativity

What do we notice here?

$$3 + 2 = 2 + 3$$
$$5 + 1 = 1 + 5$$
$$6 + 4 = 4 + 6$$

Property: $a + b = b + a$

Pose the question to the students: "Can I replace the + sign with – sign?" "Can I replace the + sign with ×?" "Can I replace the + sign with ÷?"

What do we notice here?

$$2 + (3 + 5) = (2 + 3) + 5$$
$$1 + (4 + 2) = (1 + 4) + 2$$
$$5 + (2 + 1) = (5 + 2) + 1$$

Property: $a + (b + c) = (a + b) + c$.

In a similar manner, the teacher can build patterns to demonstrate properties of multiplication and division, properties of 0 and 1 by studying the patterns.

Property: $a \times b = b \times a$, $a \times (b \times c) = (a \times b) \times c$, $a \times (b + c) = (a \times b) + (a \times c)$.

Properties of 1: $1 \times a = a$, $a \div a = 1$, $a \div 1 = a$.

Properties of 0: $a + 0 = a$, $a - 0 = a$, $a - a = 0$, $a \times 0 = 0$, $0 \div a = 0$.

ACTIVITY 7

Objective: To discover some number properties and describe them as expressions

Create a pattern with consecutive numbers.

Tell the students to sum the numbers in the pattern to discover and state the property using pattern language.

$$11 + 12$$
$$2 + 3$$
$$7 + 8$$
$$10 + 11$$

The sum of two consecutive numbers is always an odd number.

This pattern can be rewritten as follows:

11 + 12	11 + 11 + 1	2 × 11 + 1
2 + 3	2 + 2 + 1	2 × 2 + 1
7 + 8	7 + 7 + 1	2 × 7 + 1
10 + 11	10 + 10 + 1	2 × 10 + 1

It can be described as $n + n + 1$ which becomes $2n + 1$.

The students can set up patterns and discover the answers for the following questions. The answers can be stated as expressions.

What is the difference between any pair of consecutive numbers?

What is the sum of three consecutive numbers?

Can they state a property of the product of two consecutive numbers?

Can they state a property of the product of three consecutive numbers?

ACTIVITY 8

Objective: Exploring challenging problems through algebraic thinking

Let the students take any two digit number, say 53. Ask them to reverse the digits, i.e., 35. Let them find the difference between these numbers.

They can do this with some more numbers to spot a pattern.

$$53 - 35$$

$$74 - 47$$

$$21 - 12$$

$$63 - 36$$

Can they describe the pattern that emerges?

Here is one more question.

Ask the students to take a set of five numbers, say 5, 12, 4, 20, 6. Let them total it.

Now pose the following questions:

1. "If you take 2 away from each of those numbers what happens to the total? Why?"
2. "If you add 3 to each of those numbers what happens to the total? Why?"
3. "If you double each of those numbers what happens to the total? Why?"

Are they able to use expressions to answer these questions?

One final challenge!

Here is an interesting result.

$$55^2 - 45^2 = 1000$$

$$105^2 - 95^2 = 2000$$

$$85^2 - 65^2 = 3000$$

How do we describe this pattern?

Are there any other pairs which give multiples of 1000?

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box' Padmapriya may be contacted at padmapriya.shirali@gmail.com