ClassRoom

HOW to PROVE it

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In this episode of "How To Prove It," we study a number of possible characterisations of a parallelogram, as listed in the article 'Parallelogram' elsewhere in this issue.

The following question was posed in the article 'Parallelogram': What characterises a parallelogram? In other words:

> *What minimal properties must a quadrilateral have for us to know that it is actually a parallelogram?*

The basic definition of a parallelogram is: A plane four-sided figure whose opposite pairs of sides are parallel to each other. That is, a plane four-sided figure $ABCD$ is a parallelogram if and only if $AB \parallel CD$ and $AD \parallel BC$ (see Figure 1).

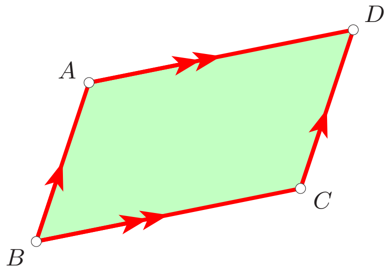


Figure 1

Here is an alternative definition, but framed in the language of transformations: A parallelogram is a quadrilateral with rotational symmetry of order 2.

Which of these properties characterises a parallelogram?

We went on to list five different properties possessed by a parallelogram and asked in each case whether the property in question characterises a parallelogram; i.e., if a planar quadrilateral possesses that property, is it necessarily a parallelogram?

Keywords: Parallelogram, characterisation, congruence, one-way implication

Note that we have retained the numbering of the items from the original article.

5. If $ABCD$ is a parallelogram, then each of its diagonals divides it into a pair of triangles with equal area. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that each of its diagonals divides it into two triangles that have equal area, is $ABCD$ necessarily a parallelogram?

6. If $ABCD$ is a parallelogram, then $AB = CD$ and $AD \parallel BC$. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that $AB = CD$ and $AD \parallel BC$, is $ABCD$ necessarily a parallelogram?

7. If ABCD is a parallelogram, then AB = CD and $\measuredangle A = \measuredangle C$. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that $AB = CD$ and $\measuredangle A = \measuredangle C$, is $ABCD$ necessarily a parallelogram?

8. If ABCD is a parallelogram, then the sum of the squares of the sides equals the sum of the squares of the diagonals. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2,$$

is $ABCD$ necessarily a parallelogram?

(9) If ABCD is a parallelogram, then the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point, is $ABCD$ necessarily a parallelogram?

The characterisations which do not work

It turns out that the statements numbered 6 and 7 are not characterisations of a parallelogram. How do we show this? In general, how do we show that any statement is not true?

Notion of a counter example. One way to disprove a statement is to exhibit a counterexample. This notion is discussed in detail in the article "Divisibility by 27," elsewhere in this issue. Nevertheless, we give a few illustrations of the notion here. Consider the following statements:

Statement 1: On observing that the odd numbers 3, 5 and 7 are prime, we may be tempted into making the following (very rash) conjecture: "All odd numbers exceeding 1 are prime." But we quickly discover a counterexample: the number 9. So the conjecture is false.

Statement 2: It is quite easy to see that if n is composite, then $2^n - 1$ is composite as well. For example, $2^{10}-1$ is a composite number (it is divisible by 3). The reader should be easily able to prove the following statement: if $n = rs$, where r and s are positive integers greater than 1 (i.e., r and s are proper divisors of n), then both $2^r - 1$ and $2^s - 1$ are proper divisors of $2^n - 1$. With this established, we may be tempted to make the following conjecture: If p is a prime number, then $2^p - 1$ is prime as well. The evidence is encouraging to start with, for the numbers

$$2^2 - 1 = 3,\ 2^3 - 1 = 7,\ 2^5 - 1 = 31,\ 2^7 - 1 = 127$$

are all prime. However, the very next number in the sequence, $2^{11} - 1$, turns out to be composite:

$$2^{11} - 1 = 2047 = 23 \times 89.$$

This means that we have found a counterexample to the stated claim, and therefore the claim is false.

Counterexample to statement 6

The question under examination is this: If $ABCD$ is a quadrilateral such that $AB = CD$ and $AD \parallel BC$, is $ABCD$ necessarily a parallelogram? The reader will readily see that the answer must be No, and that a counterexample is readily at hand; namely, an isosceles trapezium ($ABCD$ in Figure 2). Here $AD \parallel BC$, $AB = DC$ and $\measuredangle ABC = \measuredangle DCB$. (The figure may be constructed as follows. Start with a nonrectangular parallelogram *ABED*; by assumption, $\measuredangle ABE \neq 90^\circ$. There is no harm in assuming that $\measuredangle ABE < 90^\circ$. Extend *BE* and drop perpendicular *DF* to line *BE*. Extend *BF* further to *C* so that $FC = EF$. Join *DC*.)

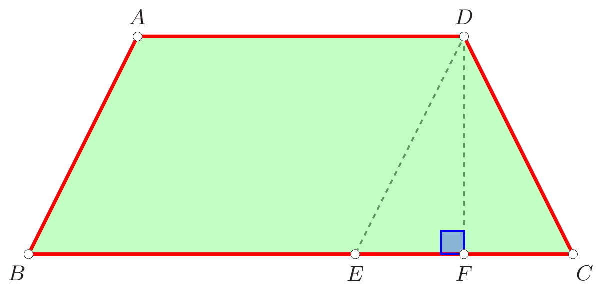


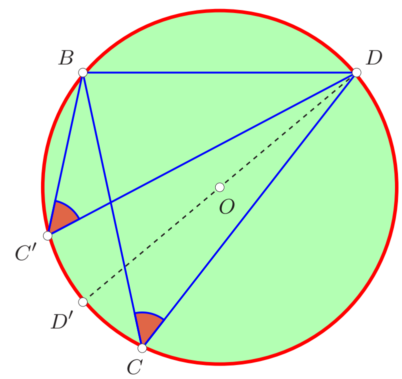
Figure 2

Counterexample to statement 7

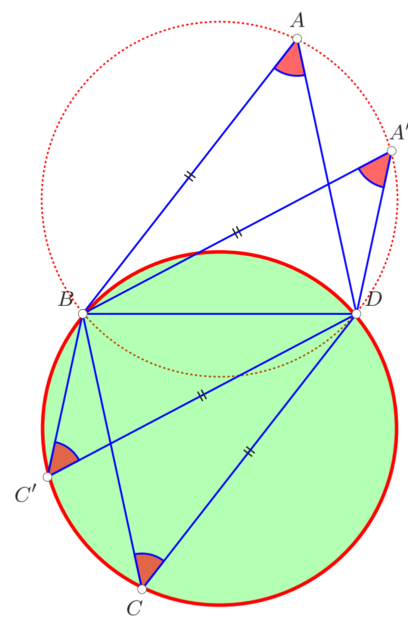
The question under examination is this: If *ABCD* is a quadrilateral such that $AB = CD$ and $\measuredangle A = \measuredangle C$, is *ABCD* necessarily a parallelogram? We shall exhibit a figure which shows that the answer is again 'No.' But finding a counterexample is more challenging now than earlier! (Please try to find one on your own before reading on.)

We make use of the symmetries of the circle. Consider the configuration shown in Figure 3 (a). It shows a chord *BD* of a circle with centre *O*; here it is important that *BD* is *not* a diameter of the circle. Infinitely many pairs of points C, C' can now be located on the circle, on the same side of *BD* as *O*, with the property that $CD = C'D$. One way to do this is to draw the diameter DD' through *D* and choose a suitable point *C* on the circle, on the same side of *BD* as *O* (a few restrictions need to be placed on the position of *C*, but we will leave it to you to work out these restrictions); then reflect *CD* in diameter DD'. Its image is $C'D$, with C' also on the circle. This does the needful. Note that in this configuration we have $CD = C'D$ and $\measuredangle BCD = \measuredangle BC'D$.

Now we locate point *A* in such a way that *ABCD* is a parallelogram as shown in Figure 3 (b). Observe now that in $ABC'D$, we have $AB = C'D$ and $\measuredangle BAD = \measuredangle BC'D$. But $ABC'D$ is clearly not a parallelogram.



(a)



(b)

Figure 3

Equally, we could locate point A' in such a way that $A'BC'D$ is a parallelogram; then $A'BCD$ has $A'B = CD$ and $\measuredangle BA'D = \measuredangle BCD$, yet it is not a parallelogram.

Remark. Points A, B, D, A' lie on a circle which is the reflection in *BD* of the circle through points B, C', C, D. This brings out an unexpected and elegant symmetry of the figure: if you rotate the entire configuration through 180° about the midpoint of *BD*, it gets mapped to itself (points B, D exchange places, as do points A, C and points A', C').

Statements 5, 8 and 9

We see that statements 6 and 7 do not provide characterisations of a parallelogram. What about statements 5, 8 and 9? They do provide the asked-for characterisations! We shall give proofs for these claims in a follow-up article.

References

1. Jonathan Halabi, "Puzzle: proving a quadrilateral is a parallelogram" from JD2718, https://jd2718.org/2007/01/10/puzzle-proving-a-quadrilateral-is-a-parallelogram/
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