ClassRoom

An Iteration on the Prime Factors of a Number

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In this short note I study the behaviour of a function f defined in the positive integers exceeding 1 (namely, the set $\{2, 3, 4, 5, \ldots\}$), when it is applied over and over again on itself. Here is its definition. Given a positive integer $n > 1$, we compute $f(n)$ as follows. First, we check whether n is prime or composite. If n is prime, then $f(n) = n + 1$. If n is composite, then we set $f(n)$ to be equal to the sum of all the prime numbers which divide n, each prime number being added as many times as it divides n. I illustrate how the definition works in Table 1.

n	Prime/composite	Prime factorisation	Computation	$f(n)$
5	prime	5	$5+1$	6
6	composite	2×3	$2+3$	5
7	prime	7	$7+1$	8
8	composite	2^3	$2+2+2$	6
9	composite	3^2	$3+3$	6
10	composite	2×5	$2+5$	7
12	composite	$2^2 \times 3$	$2+2+3$	7
20	composite	$2^2 \times 5$	$2+2+5$	9
100	composite	$2^2 \times 5^2$	$2+2+5+5$	14

Table 1

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Next, we iterate the function definition; that is, we start with some n, compute $f(n)$, then compute $f(f(n))$, then $f(f(f(n)))$, and so on, and we list the outputs in sequence. The results certainly come as a surprise; please see Table 2, where we have listed the outputs for various inputs. In every single case, the sequence ultimately settles down to . . . , 5, 6, 5, 6, 5, 6, . . .!

Starting number n	Sequence of outputs: $n, f(n), f(f(n)), \ldots$
5	5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
6	6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
7	7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
8	8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
9	9, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
10	10, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
11	11, 12, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
12	12, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
20	20, 9, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
30	30, 10, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
50	50, 12, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
100	100, 14, 9, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
1000	1000, 21, 10, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
123456	123456, 658, 56, 13, 14, 9, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .

Table 2

You will notice that in Table 2, we skipped the numbers below 5; i.e., we did not explore the output when the starting numbers are 2, 3 or 4. Table 3, below, lists the outcomes in these cases.

Starting number n	Sequence of outputs: $n, f(n), f(f(n)), \ldots$
2	2, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, . . .
3	3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, . . .
4	4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, . . .

Table 3

What these two tables show (or suggest) is that no matter what the starting number is, the outputs ultimately settle down to either the unending sequence . . . , 4, 4, 4, 4, . . ., or the unending sequence . . . , 5, 6, 5, 6, ...

How may this be explained?

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