ClassRoom

Counting Triangles

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In this article, I study the problem of counting the number of triangles formed in a triangle if n segments are drawn from one vertex to its opposite side, and h segments are drawn from another vertex to its opposite side. This kind of counting problem is often seen in puzzle collections; e.g.: "Count the number of triangles visible in Figure 1". Making a manual count for such a problem is tedious; also, it is easy to make an error in the count. We need a more analytic and systematic procedure.

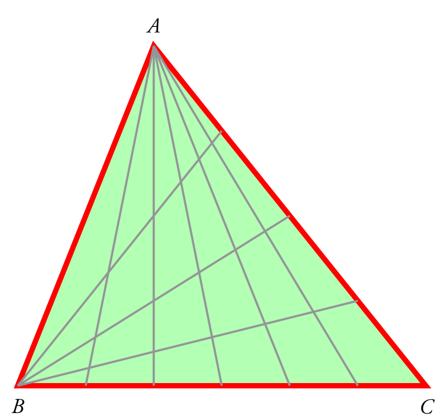

Figure 1

Keywords: *triangles, counting, combinations*

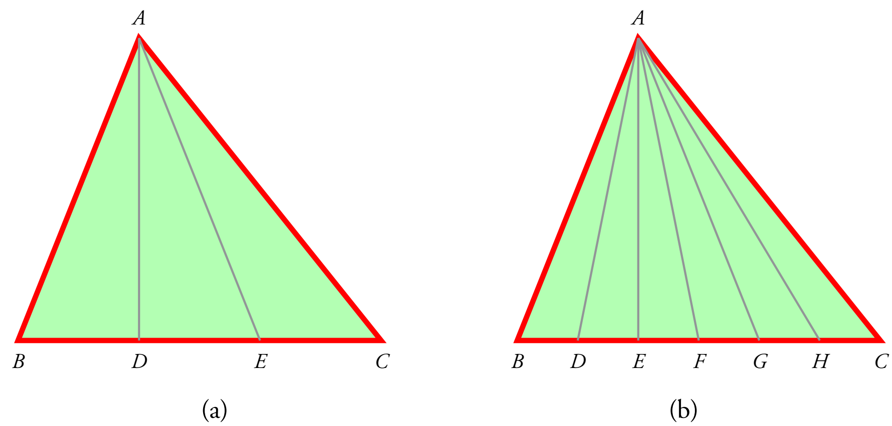

Figure 2

Step 0: Segments drawn from just one vertex
We first solve the sub-problem in which segments are drawn from just one vertex. In Figure 2 (a), lines AD, AE have been drawn from vertex A to points D, E on BC. The number of triangles thus formed can be manually counted; it comes to be 6. Now draw n segments from A to n points on BC, as in Figure 2 (b); here $n = 5$. If we take any two segments from the set of $n + 2$ segments which emanate from vertex A (i.e., the n segments along with AB and AC), we get precisely one triangle (the base lying on BC). So the number of triangles will be equal to the number of different ways of choosing 2 segments from $n + 2$ segments; this is $\binom{n+2}{2}$. Hence, for n segments drawn from a single vertex, the number of triangles is $(n + 2)(n + 1)/2$.

Step 1: Back to the original problem
Now I return to the main problem. I have divided the problem into two cases, depending on whether the number of segments emanating from the two vertices are equal or unequal.

Case 1: Same number of segments drawn from the two vertices, n = h. Consider for example $\triangle ABC$ (Figure 3), in which two segments each have been drawn from vertices B and C to the opposite sides. We first count all triangles which have B as a vertex. Triangle CBE contains $\binom{4}{2} = 6$ such triangles; likewise for triangles CBD and CBA. Therefore there are $6 + 6 + 6 = 18$ triangles which have B as a vertex. The computation may also be written as $\binom{4}{2} \times \binom{3}{1} = 6 \times 3 = 18$.

The triangles which do not have B as a vertex all lie inside $\triangle ACG$, and they all have C as a vertex. To count these, note that they can be generated by choosing any two segments out of the segments CA, CF, CG and choosing one segment from the segments BA, BD, BE. So we get $\binom{3}{2} \times \binom{3}{1} = 3 \times 3 = 9$ triangles.

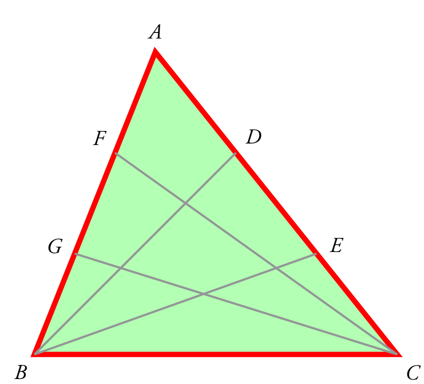

Figure 3

So, the total number of triangles is $18 + 9 = 27$.

Some of you may observe that $27 = 3^3$ which may be written as $(2+1)^3$, and guess that this is not pure coincidence!

Now we will try to prove that if n segments are drawn from both vertices, there will be $(n+1)^3$ triangles.

The proof of this general claim runs on exactly the same lines as above. Thus, in place of the quantity $\binom{4}{2} \times \binom{3}{1}$, we have the term

$$\binom{n+2}{2} \times \binom{n+1}{1};$$

and in place of the quantity $\binom{3}{2} \times \binom{3}{1}$, we have the term

$$\binom{n+1}{2} \times \binom{n+1}{1}.$$

Hence the total number of triangles in the configuration is

$$\begin{aligned}\binom{n+2}{2} \times \binom{n+1}{1} + \binom{n+1}{2} \times \binom{n+1}{1} &= \frac{(n+2)(n+1)^2}{2} + \frac{(n+1)^2 n}{2} \\ &= \frac{(n+1)^2}{2}(2n+2) \\ &= (n+1)^3.\end{aligned}$$

Case 2: Unequal number of segments drawn from the two vertices, n $\neq$ h. Now we consider the situation (Figure 4) when there are n line segments drawn from vertex B to side AC, and h line segments drawn from vertex C to side AB. Here it is assumed that $n \neq h$.

Very conveniently for us, the analysis for the general configuration runs on exactly the same lines as earlier.

Thus, in place of the term $\binom{n+2}{2} \times (n+1)$ we have the term

$$\binom{n+2}{2} \times \binom{h+1}{1};$$

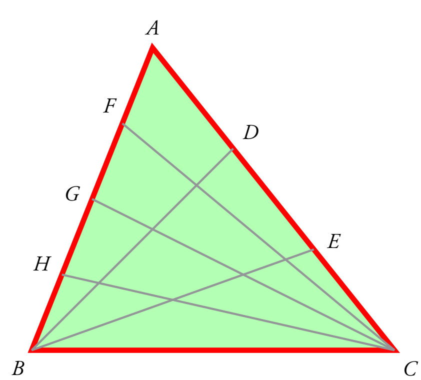

Figure 4

and in place of the term $\binom{n+1}{2} \times \binom{n+1}{1}$ we have the term

$$\binom{h+1}{2} \times \binom{n+1}{1}.$$

Hence the total number of triangles in the configuration is

$$\begin{aligned} &\binom{n+2}{2} \times \binom{h+1}{1} + \binom{h+1}{2} \times \binom{n+1}{1} \\ &= \frac{(n+2)(n+1)(h+1)}{2} + \frac{(h+1)h(n+1)}{2} \\ &= \frac{(n+1)(h+1)}{2}(n+h+2) \\ &= \frac{(n+1)(h+1)(n+h+2)}{2}. \end{aligned}$$

Remarks.

- If we interchange values of n and h, we will get the same answers by using this formula. This seems logical, as the two configurations are mirror images of each other.
- If we put $n = h$, we get the formula derived earlier, i.e., $(n+1)^3$.

Open Question

Can you find the number of triangles when n, h and k line segments are drawn from the three vertices respectively to the opposite sides? We may assume for simplicity that no three of these $n + h + k$ line segments concur.

SUNDARRAMAN MADHUSUDANAN is a grade 11 student from Mahila Samiti School and Junior College, Dombivli, Thane Dist, Maharashtra. He has been passionate about Mathematics since early childhood. The interest got kindled through the training programs conducted by the Raising a Mathematician Foundation, where he was exposed to exploration in mathematics at high school level. He loves Algebra, Discrete Mathematics and Proofs. Sundarraman wishes to pursue research in Mathematics.