An Exploration with Surds

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In this short note we present a classroom vignette involving surds. Its origin lies in the following problem.

Problem. *Simplify the following expression:*

$$\frac{1}{4+2\sqrt{3}}+\frac{1}{2\sqrt{3}+2\sqrt{2}}+\frac{1}{2\sqrt{2}+2}. \qquad (1)$$

We observe the following after the usual step of rationalising the denominators:

$$\begin{aligned}
&\frac{1}{4+2\sqrt{3}}+\frac{1}{2\sqrt{3}+2\sqrt{2}}+\frac{1}{2\sqrt{2}+2}\\
&=\frac{4-2\sqrt{3}}{16-12}+\frac{2\sqrt{3}-2\sqrt{2}}{12-8}+\frac{2\sqrt{2}-2}{8-4}\\
&=\frac{4-2\sqrt{3}+2\sqrt{3}-2\sqrt{2}+2\sqrt{2}-2}{4}\\
&=\frac{4-2}{4}=\frac{1}{2}. \qquad \text{(Note the telescopic cancellation.)}
\end{aligned}$$

It is nice to see an instance of irrational numbers adding up to a rational number!

Keywords: *Surd, irrational, telescoping sum, arithmetic progression, exploration*

Uncovering the origin of the relation. Now the exploration commences. We first try to trace the origin of this problem. Note that

$$4^2 = 16, \quad 2^2 = 4, \quad 16 - 4 = 12, \quad \frac{12}{3} = 4. \tag{2}$$

Let us divide the interval from 4 to 16 into three equal parts; we get an AP $(4,\ 8,\ 12,\ 16)$ with a common difference of 4. We write these numbers in decreasing order, from highest to lowest: $16,\ 12,\ 8,\ 4$. Now we have (note the telescopic cancellation):

$$\left(\sqrt{16} - \sqrt{12}\right) + \left(\sqrt{12} - \sqrt{8}\right) + \left(\sqrt{8} - \sqrt{4}\right) = \sqrt{16} - \sqrt{4} = 4 - 2 = 2. \tag{3}$$

From this relation we get by 'reverse-rationalisation':

$$\frac{4}{\sqrt{16} + \sqrt{12}} + \frac{4}{\sqrt{12} + \sqrt{8}} + \frac{4}{\sqrt{8} + \sqrt{4}} = 2. \tag{4}$$

The numerators in these fractions are equal precisely because $16,\ 12,\ 8,\ 4$ form an AP.

Next, the square roots of the numbers $16,\ 12,\ 8,\ 4$ are, respectively:

$$4, \quad 2\sqrt{3}, \quad 2\sqrt{2}, \quad 2.$$

Hence we get:

$$\frac{4}{4 + 2\sqrt{3}} + \frac{4}{2\sqrt{3} + 2\sqrt{2}} + \frac{4}{2\sqrt{2} + 2} = 2,$$

that is,

$$\frac{1}{4 + 2\sqrt{3}} + \frac{1}{2\sqrt{3} + 2\sqrt{2}} + \frac{1}{2\sqrt{2} + 2} = \frac{1}{2}. \tag{5}$$

We have recovered the relation with which we started the exploration.

Finding more such relations

Having uncovered the above, it becomes easy to generate more such relations. The same pair of numbers (16 and 4) can generate other APs and therefore more such relations. For example:

(1) **AP:** $16,\ 10,\ 4$**.** The common difference is 6, the square roots of the numbers are, respectively: $4,\ \sqrt{10},\ 2$, and $(4 - 2) \div 6 = 1/3$; hence we get:

$$\frac{1}{4 + \sqrt{10}} + \frac{1}{\sqrt{10} + 2} = \frac{1}{3}. \tag{6}$$

(2) **AP:** $16,\ 12,\ 8,\ 4$**.** This yields the relation we studied at the start:

$$\frac{1}{4 + 2\sqrt{3}} + \frac{1}{2\sqrt{3} + 2\sqrt{2}} + \frac{1}{2\sqrt{2} + 2} = \frac{1}{2}.$$

(3) **AP:** $16,\ 13,\ 10,\ 7,\ 4$**.** The common difference is 3, the square roots of the numbers are, respectively: $4,\ \sqrt{13},\ \sqrt{10},\ \sqrt{7},\ 2$, and $(4 - 2) \div 3 = 2/3$; hence we get:

$$\frac{1}{4 + \sqrt{13}} + \frac{1}{\sqrt{13} + \sqrt{10}} + \frac{1}{\sqrt{10} + \sqrt{7}} + \frac{1}{\sqrt{7} + 2} = \frac{2}{3}. \tag{7}$$

(4) **AP:** $16,\ 14,\ 12,\ 10,\ 8,\ 6,\ 4$**.** The common difference is 2, and $(4 - 2) \div 2 = 1$; hence:

$$\frac{1}{4 + \sqrt{14}} + \frac{1}{\sqrt{14} + 2\sqrt{3}} + \frac{1}{2\sqrt{3} + \sqrt{10}} + \frac{1}{\sqrt{10} + 2\sqrt{2}} + \frac{1}{2\sqrt{2} + \sqrt{6}} + \frac{1}{\sqrt{6} + 2} = 1. \tag{8}$$

(5) **AP:** $16,\ 15,\ 14,\ 13,\ 12,\ 11,\ 10,\ 9,\ 8,\ 7,\ 6,\ 5,\ 4$**.** This yields:

$$\frac{1}{4 + \sqrt{15}} + \frac{1}{\sqrt{15} + \sqrt{14}} + \frac{1}{\sqrt{14} + \sqrt{13}} + \cdots + \frac{1}{\sqrt{7} + \sqrt{6}} + \frac{1}{\sqrt{6} + \sqrt{5}} + \frac{1}{\sqrt{5} + 2} = 2. \tag{9}$$

The numbers on the right-hand sides of the above equalities are the following fractions whose least common denominator is 6:

$$\frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ 2, \quad \text{i.e.,} \quad \frac{2}{6},\ \frac{3}{6},\ \frac{4}{6},\ \frac{6}{6},\ \frac{12}{6}. \tag{10}$$

The numerators of these fractions are 2, 3, 4, 6, 12, which are the divisors of 12 arranged in increasing order.

For the sake of completeness, we can adjoin to the above list the fraction $1/6$ which we get as follows:

$$\frac{1}{\sqrt{16}+\sqrt{4}} = \frac{1}{6}. \tag{11}$$

We have obtained a recipé for generating such equalities!

Putting the recipé into practice

Take another pair of perfect squares, say $9 = 3^2$ and $25 = 5^2$. Since $25 - 9 = 16$, and the divisors of 16 are 1, 2, 4, 8 and 16, we obtain the following sequence of equalities:

$$\frac{1}{5+3} = \frac{1}{8},$$

$$\frac{1}{5+\sqrt{17}} + \frac{1}{\sqrt{17}+3} = \frac{2}{8},$$

$$\frac{1}{5+\sqrt{21}} + \frac{1}{\sqrt{21}+\sqrt{17}} + \frac{1}{\sqrt{17}+\sqrt{13}} + \frac{1}{\sqrt{13}+3} = \frac{4}{8},$$

$$\frac{1}{5+\sqrt{23}} + \frac{1}{\sqrt{23}+\sqrt{21}} + \frac{1}{\sqrt{21}+\sqrt{19}} + \cdots + \frac{1}{\sqrt{13}+\sqrt{11}} + \frac{1}{\sqrt{11}+3} = \frac{8}{8},$$

$$\frac{1}{5+\sqrt{24}} + \frac{1}{\sqrt{24}+\sqrt{23}} + \frac{1}{\sqrt{23}+\sqrt{22}} + \cdots + \frac{1}{\sqrt{12}+\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{10}} + \frac{1}{\sqrt{10}+3} = \frac{16}{8}.$$

Further extensions are surely possible. Please work them out on your own.

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