ClassRoom

Triangle Centres and Homogeneous Coordinates

Part I - Trilinear Coordinates

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During a course in Euclidean geometry at high school level, a student encounters four classical triangle centres—the circumcentre, the incentre, the orthocentre and the centroid (introduced as the points of concurrence of the perpendicular bisectors of the sides, the bisectors of the angles of the triangle, the altitudes and the medians, respectively). We shall study two alternative ways of describing and characterising these four significant points. They are both known as homogeneous coordinate systems, but we explain the significance of this term later. In part I of the article, we consider the first of these: trilinear coordinates.

Trilinear coordinates

This approach was suggested by the German physicist-mathematician Julius Plücker in 1835 [1]. Here a triangle centre is characterised in terms of its perpendicular distances from the three sides of the triangle; or rather, the ratios of these distances. These ratios form the "trilinear coordinates" of the triangle centre. If $\triangle ABC$ is the triangle and P the point in question, then the perpendicular distances PD, PE, PF to the sides BC, CA, AB respectively are expressed as ratios involving the side lengths and/or trigonometric functions of the angles of the triangle.

Keywords: *Triangle centre, incentre, centroid, orthocentre, circumcentre, trilinear coordinates, homogeneous coordinates, trigonometric ratio*

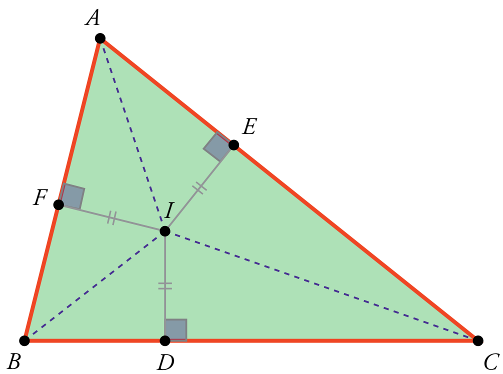

Figure 1. Incentre of a triangle; *I* is equidistant from the sides

Incentre. Since the incentre is equidistant from the sides, its trilinear coordinates are simply $1 : 1 : 1$ (see Figure 1).

Centroid. Let G be the centroid of $\triangle ABC$ (see Figure 2). It is a well-known result of Euclidean geometry that triangles GAB, GBC and GCA are equal in area. If GD, GE and GF are perpendiculars to the sides, then $GD \cdot a/2 = GE \cdot b/2 = GF \cdot c/2 = k$, say.

This yields: $GD = 2k/a$, $GE = 2k/b$, $GF = 2k/c$, hence $GD : GE : GF = 1/a : 1/b : 1/c$; these ratios form the trilinear coordinates of the centroid.

Alternatively, the coordinates could be given as $bc : ca : ab$ (multiplying through by abc), or as $\csc A : \csc B : \csc C$. The last relation arises from the fact that the sides of a triangle bear the same ratios to each other as the sines of the angles opposite them. So the reciprocals of the sides bear the same ratios to each other as the cosecant values.

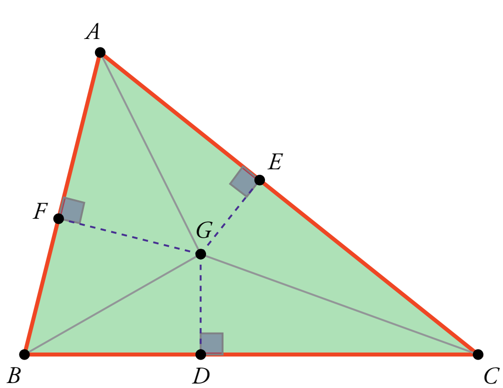

Figure 2. Centroid of an arbitrary triangle

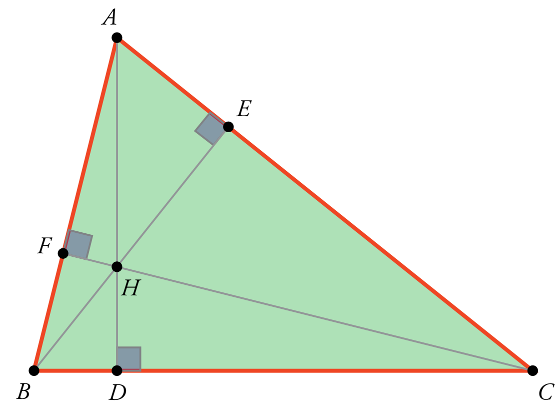

Figure 3. Orthocentre of an acute-angled triangle

Orthocentre. Now we turn our attention to the orthocentre. We first consider the case of an acute-angled triangle ABC (see Figure 3). Here, AD, BE, CF are perpendiculars from the vertices A, B, C to the sides BC, CA, AB, respectively; H is the orthocentre.

Since $\measuredangle HCD = 90^\circ - \measuredangle B$, we get $\measuredangle DHC = \measuredangle B$, and $\sec B = HC/HD$. Similarly, $\measuredangle HCE = 90^\circ - \measuredangle A$, so $\measuredangle EHC = \measuredangle A$, and $\sec A = HC/HE$. Hence:

$$\frac{HD}{HE} = \frac{HC}{HE} \Big/ \frac{HC}{HD} = \frac{\sec A}{\sec C}.$$

Similarly, $HE/HF = \sec B/\sec C$. Therefore the trilinear coordinates of the orthocentre are $\sec A : \sec B : \sec C$.

Let us see what happens as one of the angles (say $\measuredangle A$) approaches 90°. The other two angles also approach limiting values which we assume are distinct from 0° and 90°. Note that $HE/HD = \cos A/\cos B$ and $HF/HD = \cos A/\cos C$. As $\measuredangle A \to 90^\circ$, $\cos A \to 0$; so $HE \to 0$ and $HF \to 0$ (in the limit, A, H, E, F coincide; see Figure 4). Two of the three quantities HD, HE, HF are now zero, and it is customary to write the ratios as

$$HD : HE : HF = 1 : 0 : 0.$$

It follows that for a right-angled triangle ABC with $\measuredangle A = 90^\circ$, the trilinear coordinates of the orthocentre are $1 : 0 : 0$.

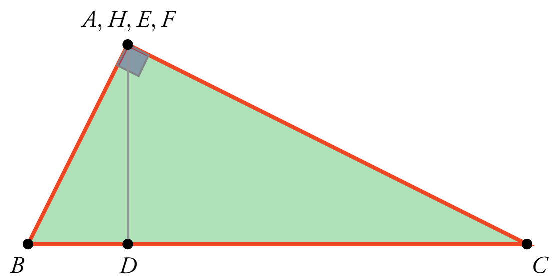

Figure 4. Orthocentre of a right-angled triangle

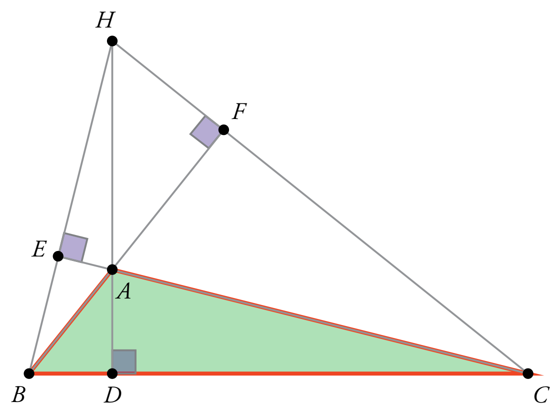

Figure 5. Orthocentre of an obtuse-angled triangle

Next, consider the case of an obtuse-angled triangle with say $\measuredangle A$ as the obtuse angle (see Figure 5); we will find that a negative sign appears in the trilinear relationship.

We have: $\measuredangle EHC = \measuredangle CAF = 180^\circ - \measuredangle A$. Then $HC/HE = \sec(180^\circ - A) = -\sec A$. Now, $HC/HD = \sec B$ (since $\measuredangle DHC = \measuredangle B$), so $HD/HE = -\sec A/\sec B$. However, $HE/HF = \sec B/\sec C$, while $HF/HD = -\sec C/\sec A$. So we get:

$$HD : HE : HF = -\sec A : \sec B : \sec C.$$

There is a way of looking at this relationship which restores the symmetry of signs. Note that if $\measuredangle A > 90^\circ$ (as in Figure 5), then H and A lie on the same side of BC, while H and B lie on opposite sides of CA, and similarly, H and C lie on opposite sides of AB. Recalling the sign convention for distances used in coordinate geometry, we see that it makes sense to regard HD as positive, and HE and HF as negative. Under this perspective, we have:

$$HD : HE : HF = \sec A : \sec B : \sec C,$$

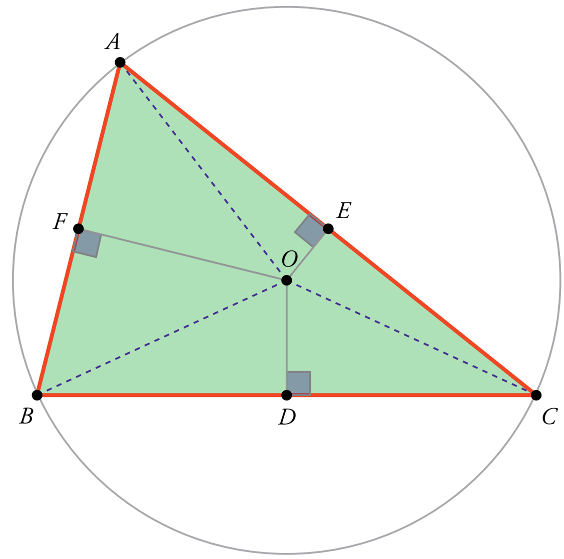

Figure 6. Circumcentre of an acute-angled triangle

and we find that the inherent symmetry of the formula has been restored.

Circumcentre. Next, we consider the circumcentre. We first look at an acute-angled triangle ABC (see Figure 6).

In the figure, O is the circumcentre of $\triangle ABC$, and OD is perpendicular to BC. We have: $\measuredangle BOC = 2\measuredangle A$, hence $\measuredangle BOD = \measuredangle A$ and $\cos A = OD/OB = OD/R$, where R is the circumradius. Thus $OD = R\cos A$. It follows that the distances from O to the sides of the triangle are proportional to the cosines of the angles opposite them. Hence the trilinear coordinates of the circumcentre are $\cos A : \cos B : \cos C$.

Just as we did last time, let us see what happens as one angle (say $\measuredangle A$) approaches 90°. The other two angles also approach limiting values which we assume are distinct from 0° and 90°.

The situation is depicted in Figure 7; D and O now coincide, and $\cos A = OD = 0$. Also

$$\frac{OE}{OF} = \frac{AB/2}{AC/2} = \frac{AB/BC}{AC/BC} = \frac{\cos B}{\cos C},$$

so $OD : OE : OF = 0 : \cos B : \cos C$. Hence the trilinear coordinates of the circumcentre are $0 : \cos B : \cos C$. This is consistent with the formula

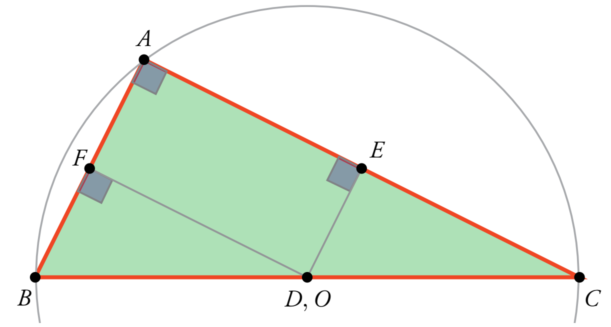

Figure 7. Circumcentre of a right-angled triangle

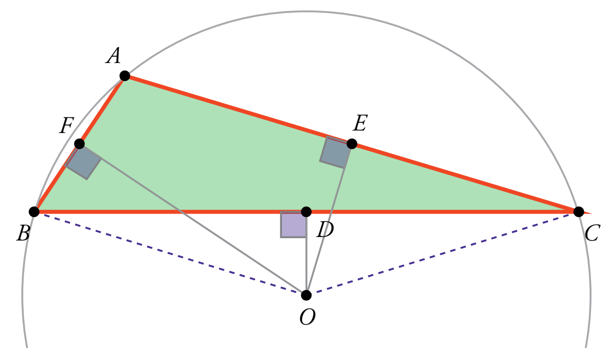

Figure 8. Circumcentre of an obtuse-angled triangle

obtained earlier; only, $\cos A$ has now assumed a zero value.

In the case of an obtuse-angled triangle, the circumcentre O lies outside the triangle (see Figure 8, where $\measuredangle A$ is obtuse).

Here we have $\measuredangle BOD = 180^\circ - A$, so $OD = -R\cos A$. The relations $OE = R\cos B$ and $OF = R\cos C$ remain unchanged. So we get $OD : OE : OF = -\cos A : \cos B : \cos C$. If we adopt the same sign convention as earlier, then $OD < 0$ since O and A lie on opposite sides of BC, while $OE > 0$ and $OF > 0$, since O and B lie on the same side of CA, and O and C lie on the same side of AB. With this understanding, the symmetry of the formula gets restored and we have: $OD : OE : OF = \cos A : \cos B : \cos C$. So the trilinear coordinates of the circumcentre are $\cos A : \cos B : \cos C$.

In Part II of the article, we shall describe another such coordinate system—barycentric coordinates.

Note from the editor: What is homogeneous about this system? The significance of the term *homogeneous* may not be immediately apparent. In 'ordinary' coordinate geometry, the equation of a line has the form $ax + by + c = 0$, where a, b, c are constants. Note that this equation is not homogeneous: two terms have degree 1, while one term has degree 0. Similarly, the question of a circle has the form $x^2 + y^2 + 2gx + 2fy + c = 0$; this too is not homogeneous. In some settings, it turns out to be advantageous to have equations which are homogeneous, in which all the terms have the same degree. The trilinear coordinates system described above has this feature, and so does the barycentric coordinates system to be discussed in part II. Here, the equation of a line has the form $lx + my + nz = 0$, where l, m, n are constants; note that this equation is homogeneous. In recent times it has been found that homogeneous coordinates are particularly convenient to use in computer graphics.

References

1. Julius Plücker. http://www-groups.dcs.st-and.ac.uk/ history/Biographies/Plucker.html
2. Wikipedia, Homogeneous Coordinates. https://en.wikipedia.org/wiki/Homogeneous_coordinates

A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.