Features

3, 4, 5 ...

And Other Memorable Triples

Part III

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In Parts I and II of this article, we studied the triples $(3, 4, 5)$ and $(4, 5, 6)$; we noted some of their properties and some geometric configurations where they occur naturally. We started with $(3, 4, 5)$ and went on to study $(4, 5, 6)$, which we nicknamed as the 'elder sibling' of $(3, 4, 5)$. In this article, we wish to study the younger sibling: the triple $(2, 3, 4)$.

But before we do that, we spend a little more time with the triple $(4, 5, 6)$. We know from Part II of the article that the triangles with sides $4, 5, 6$ has the feature that one of its angles has twice the measure of another of its angles. Also, we proved a general result:

In $\triangle ABC$ with sides a, b, c, the relation $\measuredangle A = 2\measuredangle B$ holds if and only if $a^2 = b(b + c)$.

So we start by posing the following number-theoretic problem: *Find all triples (a, b, c) of coprime positive integers satisfying the property $a^2 = b(b + c)$.* What solutions does the equation have (in coprime positive integers) other than $(a, b, c) = (6, 4, 5)$?

Keywords: *Triangle, angle, triple, ratio, double angle, observation, proof*

Solving the equation $a^2 = b(b+c)$

Note the use of the word 'coprime'. The reason for this should be clear: if a triangle with sides a, b, c has the geometrical property "one of its angles equals twice another one," then so will the triangle with sides $2a, 2b, 2c$, or one with sides $3a, 3b, 3c$; for these different triangles are all similar to one another. This statement has an exact parallel when we shift our sight to the equation $a^2 = b(b+c)$, or $a^2 = b^2 + bc$. Note that the equation is *homogeneous*: each term has degree 2. This implies that if the triple a, b, c is a solution of the equation, then so will be the triples $2a, 2b, 2c$ and $3a, 3b, 3c$; or, more generally, the triple ka, kb, kc where k is any positive integer. Hence there is no need to carry along any common factor shared by a, b, c.

There are many different ways of solving the given equation. Here is one approach. Write the equation as $a^2 - b^2 - bc = 0$, and divide through by b^2. We get:

$$\left(\frac{a}{b}\right)^2 - 1 - \frac{c}{b} = 0. \tag{1}$$

Let $u = a/b$ and $v = c/b$. Then u and v are rational numbers. The equation connecting u, v is the following:

$$u^2 - 1 - v = 0. \tag{2}$$

We must find pairs (u, v) of rational numbers that solve equation (2). This is obviously easy to do: choose any rational number $u > 1$, and compute v using the equation $v = u^2 - 1$. Then, using the values of u and v, deduce the values of a, b, c (keeping in mind the fact that they are coprime). Here are some examples:

- Take $u = 3/2$. This yields:

  $$v = \frac{9}{4} - 1 = \frac{5}{4}.$$

  Hence $a : b = 3 : 2$ and $c : b = 5 : 4$, giving $a : b : c = 6 : 4 : 5$. We have recovered the triple $(6, 4, 5)$.

- Take $u = 4/3$. This yields:

  $$v = \frac{16}{9} - 1 = \frac{7}{9}.$$

  Hence $a : b = 4 : 3$ and $c : b = 7 : 9$, giving $a : b : c = 12 : 9 : 7$. We have obtained the triple $(12, 9, 7)$. It follows that a triangle with sides $12, 9, 7$ has the property in question: one of its angles equals twice another one. (To be more specific: the angle opposite the side with length 12 is twice the angle opposite the side with length 9.)

- Take $u = 5/3$. This yields:

  $$v = \frac{25}{9} - 1 = \frac{16}{9}.$$

  Hence $a : b = 5 : 3$ and $c : b = 16 : 9$, giving $a : b : c = 15 : 9 : 16$. We have obtained the triple $(15, 9, 16)$. It follows that a triangle with sides $15, 9, 16$ has the property in question (the angle opposite the side with length 15 is twice the angle opposite the side with length 9).

- Take $u = 6/5$. This yields:

  $$v = \frac{36}{25} - 1 = \frac{11}{25}.$$

  Hence $a : b = 6 : 5$ and $c : b = 11 : 25$, giving $a : b : c = 30 : 25 : 11$. We have obtained the triple $(30, 25, 11)$. It follows that a triangle with sides $30, 25, 11$ has the property in question (the angle opposite the side with length 30 is twice the angle opposite the side with length 25).

Caution. But we obviously need to be careful when we choose a value for u. For example, suppose we choose $u = 5/2$. This yields:

$$v = \frac{25}{4} - 1 = \frac{21}{4}.$$

Hence $a : b = 5 : 2$ and $c : b = 21 : 4$, giving $a : b : c = 10 : 4 : 21$. But there clearly cannot be a triangle with sides $10, 4, 21$, because $10 + 4$ is less than 21, which means that the triangle inequality has been violated ("any two sides of a triangle are together greater than the third one").

Here is how we can resolve this problem. The triangle inequalities tell us that $a + b > c$, $b + c > a$, $c + a > b$. Also, $u = a/b$ and $v = c/b$. So in terms of u and v, we must have the following: $u + 1 > v$, $u + v > 1$, $v + 1 > u$. Or, since $v = u^2 - 1$:

$$u + 1 > u^2 - 1, \quad u + u^2 - 1 > 1, \quad u^2 - 1 + 1 > u.$$

The third condition simply tells us that $u > 1$. The second condition ($u^2 + u > 2$) is trivially satisfied if $u > 1$. So only the first condition is of relevance. It may be rewritten as $u^2 - u - 2 < 0$, i.e., $(u + 1)(u - 2) < 0$. This is true provided $-1 < u < 2$. So the three conditions together imply that $1 < u < 2$. If this condition is satisfied, we will obtain a meaningful triangle. Conversely, if the condition is not satisfied, then we obtain an "impossible" triangle. (This happens, for example, when $u = 5/2$.)

Remark. From the boundaries derived for u, we anticipate that if we choose values for u which are close to 2, we will obtain triangles which are 'thin,' i.e., with a large obtuse angle. We illustrate this remark with a numerical example. Take $u = 39/20$. This yields $a : b : c = 780 : 400 : 1121$, and the angles of the triangle are: $\measuredangle A = 25.68°$, $\measuredangle B = 12.84°$ and $\measuredangle C = 141.48°$.

Figure 1 summarises the algorithm.

Procedure for generating all coprime, positive integer triples (a, b, c) which give the sides of a triangle in which one angle is twice another

To generate all integer triples of the stated kind, we follow these steps:

- Choose a rational number u between 1 and 2.
- Compute v using the relation $v = u^2 - 1$.
- Let $(u, v) = (a/b, c/b)$ where a, b, c are positive integers and $\gcd(a, b, c) = 1$.
- Then the triangle with sides a, b, c has the required property.

Values of u which are close to 2 give 'thin' triangles with large obtuse angles.

Figure 1

The triple 2, 3, 4

We turn now to the triple $(2, 3, 4)$, the 'younger sibling' of $(3, 4, 5)$. Does it too possess some geometrical features of interest, like $(3, 4, 5)$ and $(4, 5, 6)$? Figure 2 shows a sketch of such a triangle. It has been labelled so that $a = 4$, $b = 3$, $c = 2$. Using GeoGebra, we find its angles; as we may anticipate, the triangle is obtuse-angled

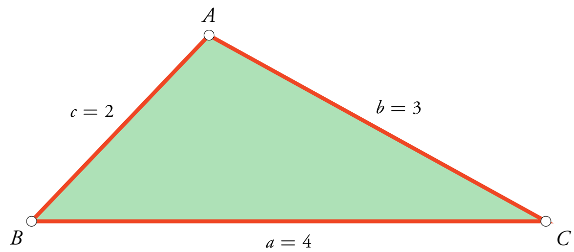


Figure 2. Triangle with sides $2, 3, 4$

(readers may recall that in Part I of this series of articles, we had proved that among all triangles whose sides are three consecutive integers, the 2-3-4 triangle is the only one which is obtuse-angled):

$$\measuredangle A = 104.48^\circ, \ \measuredangle B = 46.57^\circ, \ \measuredangle C = 28.96^\circ.$$

Examining these figures, a relationship connecting them does not immediately strike the eye. But we do find, after some searching, a curious relationship between them, prompted perhaps by the equality $96 = 2 \times 48$. We have:

$$\measuredangle A - 90^\circ = 14.48^\circ, \quad \measuredangle C = 28.96^\circ,$$

and $28.96 = 2 \times 14.48$, i.e., $\measuredangle C = 2\left(\measuredangle A - 90^\circ\right)$. Well, that is something! There is, after all, a relationship of note between the angles of the triangle.

Just as we did when we discovered a certain relationship between the angles of the triangle with sides $4, 5, 6$, we need to ensure that this observed relationship is exact and not approximate. (It could just be the case that equality holds till ten decimal places but not beyond that $\ldots$.) Once again, we opt for a trigonometric proof of the equality.

The observed relationship may be rewritten as $\measuredangle C = 2\measuredangle A - 180^\circ$, which implies that $\cos C = -\cos 2A$. Does the reverse implication hold? That is, does the equality $\cos C = -\cos 2A$ imply that $\measuredangle C = 2\measuredangle A - 180^\circ$? We already know that $\measuredangle C$ is acute (because c is the smallest side), while $\measuredangle A$ is obtuse (because $a^2 > b^2 + c^2$). Now assume that $\cos C = -\cos 2A$. Then we can be sure that at least one of the following statements is true:

(1) $\measuredangle C + 2\measuredangle A$ is an odd multiple of 180°;

(2) $\measuredangle C - 2\measuredangle A$ is an odd multiple of 180°.

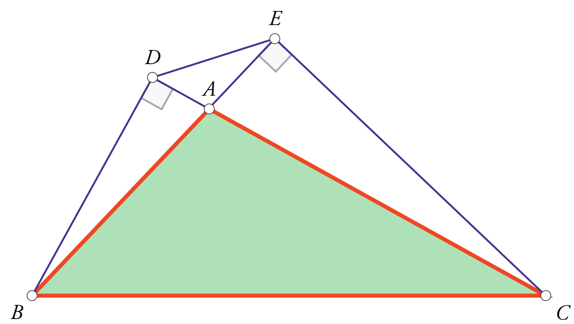
Of these, statement (a) is not possible; for if $0^\circ < \measuredangle C < 90^\circ$ and $90^\circ < \measuredangle A < 180^\circ$, then $180^\circ < \measuredangle C + 2\measuredangle A < 450^\circ$. Hence statement (b) must be true. But which odd multiple of 180° is $\measuredangle C - 2\measuredangle A$ equal to? Since $-360^\circ < \measuredangle C - 2\measuredangle A < -90^\circ$, it must be equal to -180°. So we only need to establish that $\cos C = -\cos 2A$. We have:

$$\cos C = \frac{3^2 + 4^2 - 2^2}{2 \times 3 \times 4} = \frac{7}{8},$$

$$\cos A = \frac{2^2 + 3^2 - 4^2}{2 \times 2 \times 3} = -\frac{1}{4},$$

$$\cos 2A = 2\cos^2 A - 1 = \frac{2}{16} - 1 = -\frac{7}{8}.$$

We see that $\cos C = -\cos 2A$, and it follows that $\measuredangle C = 2\measuredangle A - 180^\circ$. Hence the observed relationship holds exactly.



- $AD = 2k$
- $AE = 3k$
- $DE = 4k$
- $BE = 2 + 3k$
- $CD = 3 + 2k$

Figure 3. Demonstrating that $\measuredangle BED$ is twice $\measuredangle DBE$

A geometric interpretation

The angle relationship we have proved can also be illustrated and proved geometrically. Figure 3 shows the 2-3-4 triangle with sides BA and CA extended beyond vertex A, and perpendiculars BD and CE drawn from vertices B and C to the extended sides. Observe that quadrilateral $DBCE$ is cyclic (because $\measuredangle BDC$ and $\measuredangle BEC$ are both right angles; so BC is a diameter of the circumcircle of $DBCE$). In this figure we have $\measuredangle DBE = \measuredangle DCE$. The angle relationship $\measuredangle C = 2\measuredangle A - 180^\circ$ is equivalent to stating that each of $\measuredangle DBE$ and $\measuredangle DCE$ is equal to half of $\measuredangle BCD$ (for we have: $\measuredangle DBE = \measuredangle A - 90^\circ = \measuredangle DCE$). But we also have $\measuredangle BED = \measuredangle BCD$, by the property of a cyclic quadrilateral. Hence the stated property is equivalent to the following: *In $\triangle BED$, we have $\measuredangle BED = 2\measuredangle DBE$.*

The last statement should make us prick up our ears: it connects the property currently under study with what we studied in the previous part of this article (in the July 2015 issue of *At Right Angles*). We had earlier established the conditions under which one angle of a triangle is twice another angle of the same triangle. Invoking that result, we see that the desired angle relationship will be established if we show that $BD^2 = DE(DE + BE)$. This is what we now do.

As quadrilateral $DBCE$ is cyclic, $\triangle ABC$ is similar to $\triangle ADE$ (see Figure 3). Let the ratio of similarity be $1 : k$. Since the sides of $\triangle ABC$ are 2, 3, 4, the sides of $\triangle ADE$ will be $2k, 3k, 4k$. Since $\triangle ABD$ is right-angled at D, we get by Pythagoras's theorem:

$$BD^2 = 2^2 - (2k)^2 = 4\left(1 - k^2\right).$$

From $\triangle BDC$, which too is right-angled at D, we get $BD^2 + CD^2 = BC^2$, hence:

$$4\left(1 - k^2\right) + (3 + 2k)^2 = 4^2.$$

Solving this equation for k, we get $k = 1/4$. Hence:

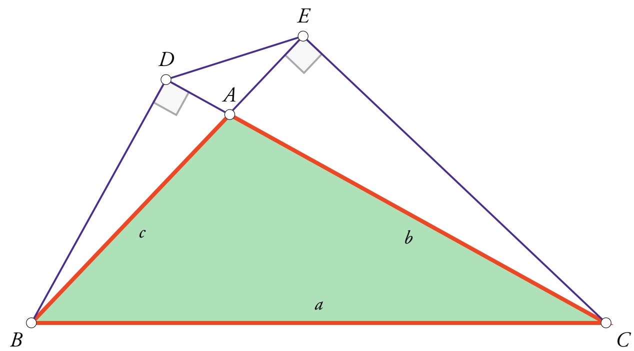
$$BD^2 = \frac{15}{4}, \quad DE = 1, \quad BE = 2 + \frac{3}{4} = \frac{11}{4},$$

$$DE(DE + BE) = \frac{15}{4},$$

and we see that $BD^2 = DE(DE + BE)$. It follows that $\measuredangle BED$ is twice $\measuredangle DBE$. The required property has thus been proved.

The general condition

Now we ask the following question: what condition must be placed on the sides a, b, c of $\triangle ABC$ so that it satisfies a property of the kind studied above? That is (see Figure 4), $\measuredangle BAC$ must be obtuse, and when perpendiculars BD and CE are drawn from vertices B and C to the extended sides CA and BA respectively, we must have: $\measuredangle ACB = 2\measuredangle DBA$.



- $AD = ck$
- $AE = bk$
- $DE = ak$
- $BE = c + bk$
- $CD = b + ck$

Figure 4. Under what conditions is it true that $\measuredangle BED$ is twice $\measuredangle DBE$?

We now ask: what conditions must a, b, c satisfy in order that $BD^2 = DE(DE + BE)$. We can opt either for a trigonometric approach now, or a pure geometry approach. We take the latter path.

As quadrilateral $DBCE$ is cyclic, $\triangle ABC \sim \triangle ADE$. Let the ratio of similarity be $1 : k$, as earlier. Since the sides of $\triangle ABC$ are a, b, c, the sides of $\triangle ADE$ will be ak, bk, ck. Since $\triangle ABD$ is right-angled at D, we get by Pythagoras's theorem:

$$BD^2 = c^2 - (ck)^2 = c^2\left(1 - k^2\right).$$

From $\triangle BDC$, which too is right-angled at D, we get $BD^2 + CD^2 = BC^2$, hence:

$$c^2\left(1 - k^2\right) + (b + ck)^2 = a^2.$$

Solving this equation for k, we get (algebraic details omitted):

$$k = \frac{a^2 - b^2 - c^2}{2bc}.$$

This yields (algebraic details omitted yet again; as the reader may have guessed, these algebraic computations have been done using a computer algebra system; I would not dare to go through this kind of algebra using hand calculation alone!):

$$DE = \frac{a\left(a^2 - b^2 - c^2\right)}{2bc},$$

$$BE = \frac{a^2 - b^2 + c^2}{2c},$$

and therefore:

$$DE + BE = \frac{a^3 + a^2 b - a\left(b^2 + c^2\right) - b\left(b^2 - c^2\right)}{2bc},$$

$$BE(DE + BE) = \frac{a\left(a^2 - b^2 - c^2\right)\left(a^3 + a^2 b - a(b^2 + c^2) - b(b^2 - c^2)\right)}{4b^2c^2},$$

$$BD^2 = \frac{(a + b + c)(a + b - c)(b + c - a)(c + a - b)}{4b^2}.$$

Note that the expression for BD can be obtained without needing to use the expression for k. We only need to see that $BD \times b/2$ is equal to the area of $\triangle ABC$, and an expression for the area is known from Heron's formula.

On equating the expressions for $BE(DE+BE)$ and BD^2 and going through the algebra, we obtain the following condition: $BE(DE+BE) = BD^2$ is true if and only if $a > b$ and

$$a^2 - ab - c^2 = 0.$$

We have obtained the conditions which will ensure the desired property. It is easily checked that $(a, b, c) = (4, 3, 2)$ satisfies the conditions.

Generating integer triples that satisfy the property

Just as we did earlier, we now ask for a way of generating more coprime integer triples for which the desired geometrical property holds. It turns out that the approach followed earlier works here as well.

We need to find integer triples (a, b, c) for which $a > b$ and $a^2 = ab + c^2$. Note that this relation automatically ensures that a is the largest side, and $\measuredangle A$ is obtuse. By dividing through by b^2, we may rewrite the relation as:

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b} + \left(\frac{c}{b}\right)^2.$$

Let $u = a/b$ and $v = c/b$. Naturally, u and v are positive rational numbers, with $u > 1$, $u > v$. The above relation takes the following form:

$$u^2 = u + v^2, \qquad \therefore\ u(u-1) = v^2.$$

To generate solutions to the equation $u(u-1) = v^2$, we adopt the following artifice. We write the above relation as:

$$\frac{u}{v} = \frac{v}{u-1},$$

and denote the value of u/v by t. We then have:

$$\begin{cases} u = tv, \\ v = t(u-1). \end{cases}$$

Treating t as a parameter (note that $t > 1$), we solve these two equations simultaneously for u and v. We get (algebraic details omitted):

$$u = \frac{t^2}{t^2-1}, \qquad v = \frac{t}{t^2-1}.$$

Hence we have $u : 1 : v = t^2 : t^2 - 1 : t$, i.e.,

$$a : b : c = t^2 : t^2 - 1 : t.$$

This parametrisation allows us to generate infinitely many integer triples (a, b, c) which satisfy the desired property. For example:

- Take $t = 2$. We get $a : b : c = 4 : 3 : 2$. This yields the very triangle we have been studying.
- Take $t = 3$. We get $a : b : c = 9 : 8 : 3$. It may be checked that the triangle with sides 9, 8, 3 possesses the property in question: its angles are 99.59407°, 61.2178° and 19.18814°, and we have: $19.18814 = 2 \times (99.59407 - 90)$.
- Take $t = 4$. We get $a : b : c = 16 : 15 : 4$. It may be checked that the triangle with sides 16, 15, 4 possesses the property in question: its angles are 97.18076°, 68.45773° and 14.36151°, and we have: $14.36151 = 2 \times (97.18076 - 90)$.

Figure 5 summarises the algorithm.

Procedure for generating all coprime, positive integer triples (a, b, c) which give the sides of a triangle in which $\measuredangle C = 2\,(\measuredangle A - 90°)$

To generate all integer triples of the stated kind, we follow these steps:

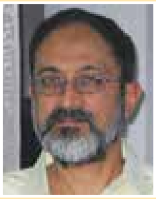
- Choose a rational number $t > 1$.
- Compute u and v using the relations

$$u = \frac{t^2}{t^2 - 1}, \quad v = \frac{t}{t^2 - 1}.$$

- Let $u : 1 : v = a : b : c$ where a, b, c are positive integers and the gcd of a, b, c is 1.
- Then the triangle with sides a, b, c has the required property.

Figure 5

What about the triangle inequality? We may wonder whether some restrictions have to be placed on t for the triangle inequality to be satisfied, i.e., for us to get a valid triangle. But in this case, unlike the previous one, the problem resolves itself on its own: any value of t which exceeds 1 will suffice. For if $t > 1$, we have $t^2 > t$; and $t^2 > t^2 - 1$ is always true; so the inequality to be checked is: $(t^2 - 1) + t > t^2$. But this is automatically satisfied, since $t > 1$.



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