Features

At Right Angles!

Integer-Sided Triangles with Perpendicular Medians

...and how we got there

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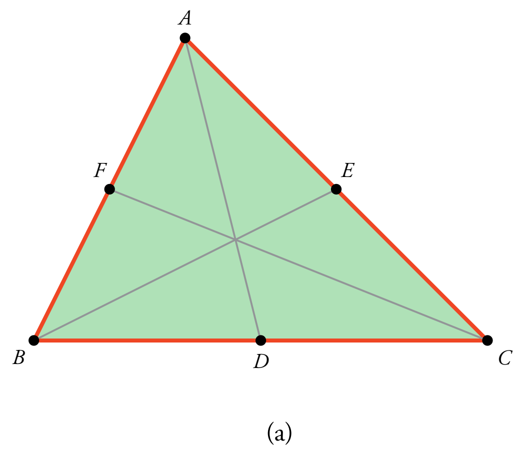
In this note we discuss the conditions that must be satisfied by the sides of an arbitrary integer-sided triangle if its medians can serve as the sides of a right-angled triangle.

It is well known (and easy to prove) that given any triangle ABC, there exists a triangle whose three sides are respectively congruent to the three medians of $\triangle ABC$. This triangle is sometimes called the ***median triangle*** of $\triangle ABC$. (Note that this is not the same as the *medial triangle*, which is the triangle whose vertices are the midpoints of the sides of the triangle. The two notions must not be confused with each other.) In this note, we ask for the condition that must be satisfied by the sides of $\triangle ABC$ in order that its median triangle be right-angled. After obtaining the condition, we consider the problem of generating integer triples (a, b, c) that satisfy this condition.

Lemma 1. *Let $\triangle ABC$ is an arbitrary triangle with sides a, b, c. If m_a, m_b, m_c are the lengths of the medians drawn to the sides BC, CA, AB respectively, then there exists a triangle whose sides have lengths m_a, m_b, m_c. That is, if ABC is any triangle, another triangle can always be constructed whose sides are equal to the medians of $\triangle ABC$.*

Keywords: *Pythagoras theorem, Apollonius theorem, triangle, median, Diophantine equation*



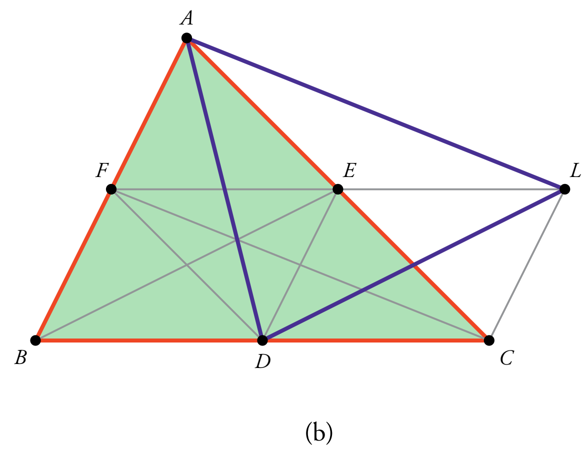


Figure 1. Construction of the median triangle ($\triangle ADL$) of $\triangle ABC$; note that its sides DL and LA are parallel to the medians BE and CF respectively

Proof. Our proof is constructive: we show how to actually construct the median triangle of ABC. Let AD, BE, CF be the medians of $\triangle ABC$; see Figure 1 (a). Through D draw DL equal and parallel to BE; see Figure 1 (b). Join AL; then we claim that $\triangle ADL$ is the required triangle.

For proof, we only have to establish that AL is equal and parallel to FC. We show this as follows. Since $EBDL$ is a parallelogram (by construction), EL is equal and parallel to BD, hence EL is equal and parallel to DC. This implies that $EDCL$ is a parallelogram; so LC is equal and parallel to ED. From this it follows that LC is equal and parallel to AF. Hence $AFCL$ is a parallelogram and AL is equal and parallel to FC, as required. Therefore the sides of $\triangle ADL$ are respectively equal and parallel to the medians AD, BE, CF. $\square$

Lemma 2 (Apollonius)**.** *If m_a, m_b, m_c are the lengths of medians AD, BE, CF of $\triangle ABC$ drawn to the sides $BC = a$, $CA = b$, $AB = c$, then m_a, m_b, m_c are given by:*

$$\begin{aligned} 4m_a^2 &= 2b^2 + 2c^2 - a^2, \\ 4m_b^2 &= 2c^2 + 2a^2 - b^2, \\ 4m_c^2 &= 2a^2 + 2b^2 - c^2. \end{aligned}$$

This is simply a statement of the theorem of Apollonius, and it follows from the application of the Pythagorean theorem to suitably constructed triangles.

A right-angled median triangle

We now ask the following question:

> *What conditions must a, b, c satisfy if the median triangle is to be right-angled?*

In view of the proof of Lemma 1, the above question is equivalent to asking the following question: *What conditions must a, b, c satisfy if some two medians of $\triangle ABC$ are to be perpendicular to each other?*

Referring to Figure 1 (b), let us require (without any loss of generality) that $\triangle ADL$ be right-angled at vertex D, i.e., that $\angle ADL$ is a right angle. (This is equivalent to requiring that the medians through A and B are perpendicular to each other.) In this case we must have $AD^2 + DL^2 = AL^2$, i.e.,

$$m_a^2 + m_b^2 = m_c^2. \tag{1}$$

Using Lemma 2, this may be rewritten as:

$$\begin{aligned} &\left(2b^2 + 2c^2 - a^2\right) + \left(2c^2 + 2a^2 - b^2\right) \\ &\quad = 2a^2 + 2b^2 - c^2. \end{aligned} \tag{2}$$

This in turn simplifies to the following condition:

$$a^2 + b^2 = 5c^2. \tag{3}$$

It is easy to check that the converse holds as well, i.e., if (3) is true, then so is (1), implying that $\triangle ADL$ is right-angled at D. So we have obtained the required condition: given an arbitrary triangle ABC, the medians through A and B are perpendicular to each other if and only if $a^2 + b^2 = 5c^2$.

Finding integer solutions to the resulting condition

Having obtained the required condition on the triple (a, b, c), we now ask: *How do we generate the family of integer solutions to this equation?* Here is one approach which helps in obtaining this family. We write the equation in the following form:

$$a^2 + b^2 = 4c^2 + c^2. \tag{4}$$

We know that the following relation is an identity, true for all p, q, m, n:

$$(pm + qn)^2 + (pn - qm)^2 = (pm - qn)^2 + (pn + qm)^2. \tag{5}$$

Looking closely at (4) and (5), let us put:

$$a = pm + qn,$$

$$b = pn - qm,$$

$$2c = pm - qn,$$

$$c = pn + qm.$$

From the last two equations we get $2(pn + qm) = pm - qn$, hence $p(m - 2n) = q(2m + n)$, i.e.,

$$\frac{p}{q} = \frac{2m + n}{m - 2n}.$$

Therefore let us write:

$$p = k(2m + n), \tag{6}$$

$$q = k(m - 2n), \tag{7}$$

where k is some rational number, suitably chosen. We now obtain, on substitution:

$$a = pm + qn = k\left(2m^2 + 2mn - 2n^2\right),$$

$$b = pn - qm = k\left(-m^2 + 4mn + n^2\right),$$

$$c = pn + qm = k\left(m^2 + n^2\right).$$

It follows that the triple

$$(a, b, c) = \Big(k\left(2m^2 + 2mn - 2n^2\right), k\left(-m^2 + 4mn + n^2\right), k\left(m^2 + n^2\right)\Big) \tag{8}$$

satisfies the condition $a^2 + b^2 = 5c^2$. We must choose k so that a, b, c turn out to be integers.

Some restrictions are needed on m, n in order for us to get a valid triangle: the sides of the triangle must be positive, and the three triangle inequalities must be satisfied (the sum of any two sides must exceed the third side). Hence we must have:

$$a > 0,\ \ b > 0,\ \ c > 0,\ \ a + b > c,$$
$$b + c > a,\ \ c + a > b.$$

That is:

$$m^2 + mn - n^2 > 0, \quad m^2 - 4mn - n^2 < 0, \quad m^2 + n^2 > 0, \tag{9}$$

and:

$$3mn - n^2 > 0, \quad m^2 - mn - 2n^2 < 0, \quad 2m^2 - mn - n^2 > 0. \tag{10}$$

To see what these inequalities lead to, it becomes simpler if we divide each one by n^2 and write $t = m/n$. Here is what we get (we have divided out by 2 in some cases):

Inequality	Solution set
$t^2 + t - 1 > 0$	$t < -\frac{1}{2}\left(\sqrt{5} + 1\right)$ or $t > \frac{1}{2}\left(\sqrt{5} - 1\right)$
$t^2 - 4t - 1 < 0$	$2 - \sqrt{5} < t < 2 + \sqrt{5}$
$t^2 + 1 > 0$	Always true
$3t - 1 > 0$	$t > \frac{1}{3}$
$t^2 - t - 2 < 0$	$-1 < t < 2$
$2t^2 - t - 1 > 0$	$t < -\frac{1}{2}$ or $t > 1$

The interval common to all the solution sets is: $1 < t < 2$. Hence the conditions on m, n are:

$$m > 0, \quad n > 0, \quad n < m < 2n. \tag{11}$$

With these conditions satisfied, any triple (a, b, c) computed using the following formula:

$$(a, b, c) = \Big(k\left(2m^2 + 2mn - 2n^2\right), k\left(-m^2 + 4mn + n^2\right), k\left(m^2 + n^2\right)\Big), \tag{12}$$

k being any rational number such that a, b, c are integers, can serve as the sides of an integer-sided triangle for which the medians through vertices A and B are perpendicular to each other. Note that k can be of the form $1/r$, where r is the gcd of $2m^2 + 2mn - 2n^2$, $-m^2 + 4mn + n^2$ and $m^2 + n^2$.

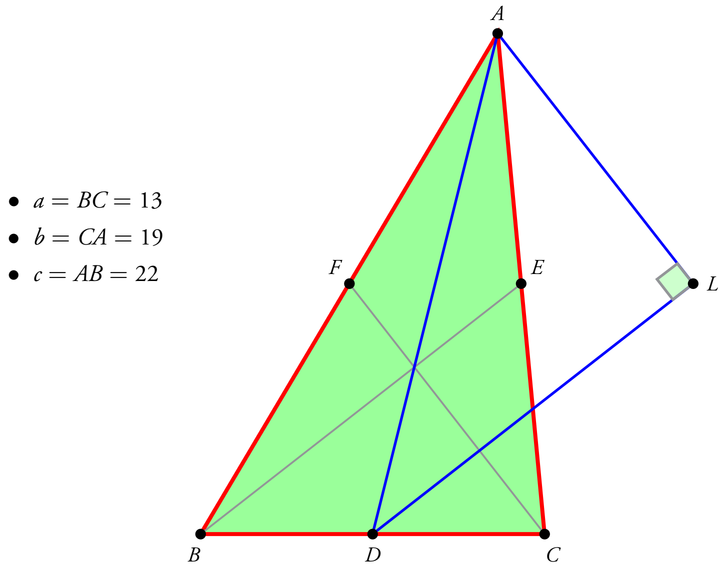


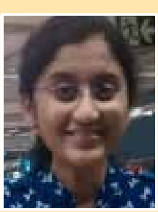
Figure 2. Triangle with sides $22, 19, 13$ and its median triangle; here $BE \perp CF$

Two examples are given below where k is not an integer.

Examples of some triples.

- Let $m = 3$, $n = 2$, $k = 1$; we get:
$$(a, b, c) = (22, 19, 13).$$
- Let $m = 4$, $n = 3$, $k = 1$; we get:
$$(a, b, c) = (38, 41, 25).$$
- Let $m = 5$, $n = 3$, $k = 1/2$; we get:
$$(a, b, c) = (31, 22, 17).$$
- Let $m = 12$, $n = 11$, $k = 1/5$; we get:
$$(a, b, c) = (62, 101, 53).$$

Figure 2 shows a triangle with sides 22, 19, 13 respectively, and its associated right-angled median triangle.



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