Triangle Equalizers

A Probabilistic Approach

Part I

Features

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An equalizer of a triangle is a line which divides it into two regions having equal areas as well as equal perimeters. Every triangle has at least one and at most three equalizers. In [1], triangles of all three types are identified and it is remarked that triangles with two equalizers are quite rare. In [2], an example of such a triangle is given. In this two-part note, we simplify the results of [1]. In particular, we show that in a certain probabilistic sense, the probability that a triangle taken at random has two equalizers is 0. The method is based on the roots of quadratic equations, a technique already initiated in [2]. In Part II, we estimate the probabilities that a triangle taken at random has a given number of equalizers. In particular, we show that more than 85% of the triangles have only one equalizer. Using *Mathematica*, the figure is found to be close to 98%.

Introduction and Terminology

As usual, given a triangle ABC, we denote its sides opposite to A, B, C by a, b, c respectively, its semi-perimeter $\frac{1}{2}(a+b+c)$ by s and its incentre by I. Suppose a line ℓ is an equalizer of ABC. Then it is easy to show (see, e.g. [2]) that ℓ passes through I. Consequently, if ℓ passes through some vertex, say A, of ABC, then it must be the line AI whence the triangle must be isosceles with $AB = AC$. In all other cases, an equalizer must cut two of the three sides internally. Without loss of generality, we take these sides to be AB and AC and denote their points of intersection with ℓ by X and Y respectively. We say that this equalizer is *opposite* to the side a. (See Figure 1.)

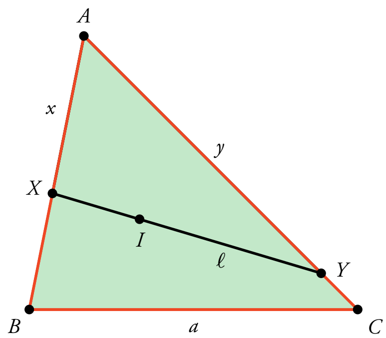

Figure 1. Equalizer opposite to a

Since a triangle can have at most three equalizers, the probability that a randomly taken line passing through the incentre of a triangle is an equalizer of it is 0. But the problem we address here is that of determining with what probabilities a randomly taken triangle has 1, 2 or 3 equalizers.

We first consider the problem of determining the conditions under which ABC has an equalizer opposite to a. For this we let $x = AX$ and $y = AY$. Then by definition of an equalizer, $x + y = s$ and $xy = \frac{1}{2}bc$. These equations together imply that x and y are the roots of the quadratic polynomial

$$f(p) = p^2 - sp + \frac{1}{2}bc. \qquad (1)$$

A direct calculation gives

$$f(0) = f(s) = \frac{bc}{2},$$
$$f(a) = \frac{(b-a)(c-a)}{2}, f(b) = \frac{b(b-a)}{2},$$
$$f(c) = \frac{c(c-a)}{2}. \qquad (2)$$

The existence of an equalizer opposite to a is equivalent to $f(p)$ having two (possibly equal) roots, one lying in the interval $(0, b)$ and the other in $(0, c)$. This observation, coupled with the equations above, enables us to determine the number of equalizers opposite to a depending upon how a compares with b and c. To avoid degeneracies, we first consider only scalene triangles, i.e., triangles ABC where a, b, c are distinct.

Numbers of Equalizers in Scalene Triangles

We assume that a, b, c are all distinct. We consider three cases depending upon where a is placed compared with b and c.

(i) Suppose first that a is the longest side. Then both $f(b)$ and $f(c)$ are negative. As the leading coefficient of $f(p)$ is positive, this means that b, c lie between the two roots of $f(p)$. Hence the larger of the two roots is bigger than both b, c, violating the requirement of an equalizer that one root must be less than b and the other one less than c. So in this case, though the quadratic (1) has real roots, there is no equalizer opposite to a.

(ii) Assume that a lies between b and c. Without loss of generality, assume that $b < a < c$. Then $f(0) > 0, f(b) < 0$ and $f(c) > 0$. So, by continuity of the quadratic function, f has at least one root in the interval $(0, b)$ and at least one root in the interval (b, c). As there are only two roots, there is exactly one root in $(0, b)$ and one in (b, c). So there is exactly one equalizer opposite to a, viz. the line XY with x representing the larger and y the smaller root. (Note that the roots cannot be interchanged because of the inequalities they must satisfy.)

(iii) Assume that a is the smallest side. This is the most interesting case. The quadratic (1) may not have any real roots in this case, as happens, for example, when $(a, b, c) = (3, 4, 5)$. But we claim that if at all (1) has a root, then every root gives rise to an equalizer opposite to a. (In [2], this is mentioned only as a possibility. We claim that it is a certainty.)

So assume that $s^2 \geq 2bc$, and let x, y be the roots of $f(p)$ with $x \leq y$. Without loss of generality, assume that $a < b < c$. Here $f(a), f(b), f(c)$ are all positive, so mere continuity of the quadratic is not of much help. Here we use the fact that the graph of $f(p)$ is a parabola with its lowest point at $p = s/2$ and symmetric about the line $p = s/2$. Figure 2 shows a sketch of the graph for

$0 \leq p \leq s$. The portion from 0 to $s/2$ is strictly decreasing, while that between $s/2$ and s is strictly increasing. Note further that x lies in the left half and y in the right half, i.e. $x \leq s/2$ and $s/2 \leq y$. (If $s^2 = 2bc$, then both the roots equal $s/2$, and the parabola touches the p-axis. This happens, for example, when $(a, b, c) = (7, 8, 9)$.)

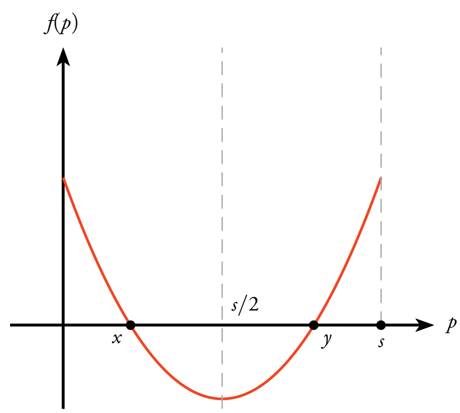

Figure 2. Graph of the quadratic $f(p)$

Since $f(b) > 0$, b cannot lie in the interval $[x, y]$. We claim that b lies to the right of this interval, i.e. that both x and y are less than b (and hence c too). For, if this is not so, then b and hence a too must lie to the left of x. But as f is strictly decreasing on $[0, x]$ this would mean $f(a) > f(b)$. But on the other hand $f(a) < f(b)$, a contradiction.

Thus we have shown that if at all (1) has a root, then both roots lie in the interval $(0, b)$ and hence in the interval $(0, c)$ too. If they are equal, there will be only one equalizer XY opposite to a. Moreover, it will be perpendicular to the angle bisector AI because $\triangle AXY$ is isosceles. But if the roots x and y are distinct, then XY will be an equalizer opposite to a and so will be $X'Y'$ where X', Y' are points on AB, AC at distances y and x respectively from A. (As already noted, a similar interchange was not possible in (ii) above.)

Summing up, we have shown that a scalene triangle has no equalizer opposite to its longest side, one equalizer opposite to its middle side, and either 0, 1 or 2 equalizers opposite to its shortest side. These three possibilities occur according as s^2 is less than, equal to or greater than, twice the product of the two longer sides. The three possibilities are illustrated by the triangles with sides $(13, 16, 19)$, $(14, 16, 18)$ and $(15, 16, 17)$ respectively, with s being 24 in each case.

The reasoning here can be modified to include cases of equality of some pairs of sides. Suppose, for example, that $\triangle ABC$ is isosceles with $AB = AC > BC$. Then a is the shortest side. Both b and c can be considered as the middle side if we drop the strictness of inequalities, because we have $a < b \leq c$ and also $a < c \leq b$. In this case, one root of the quadratic $p^2 - sp + \frac{1}{2}ab$ is b, and the other root is $\frac{1}{2}a$. So the equalizer opposite to b is simply the median through A. This is also the equalizer opposite to c.

Triangles with Two Equalizers

The above analysis provides a complete characterization of triangles having exactly two equalizers.

Theorem 1. *A triangle ABC has exactly two equalizers if and only if one of the following two possibilities holds:*

(i) *$\triangle ABC$ is scalene, and s^2 equals twice the product of the two longer sides.*

(ii) *$\triangle ABC$ is isosceles, and its unequal angle is equal to $2\sin^{-1}(\sqrt{2} - 1)$.*

Proof. The case of a scalene triangle was already considered above. An equilateral triangle has (at least) three equalizers; namely, its three medians. (It is easy to show that it cannot have any others. But that is not needed here.) So the only possibility left is one where $\triangle ABC$ is isosceles with $AB = AC \neq BC$ (say). Here we have $b = c$ and $a \neq b$. The median through A is an equalizer. It operates simultaneously as an equalizer opposite to both b and c. So, the second equalizer, if any, must be opposite to a. If $a > b$, then this is impossible; the reasoning is similar to case (i) of the scalene triangles, because in this case too, (1) cannot have any roots less than b (or c).

It remains to deal with the case where $b = c > a$. Here too, the argument in case (iii) for scalene triangles goes through and shows that there is a unique equalizer opposite to a if and only if $s^2 = 2bc = 2b^2$, i.e. if and only if $s = \sqrt{2}b$. But, on the other hand, $s = b + \frac{1}{2}a$. Hence:

$$\frac{a}{b} = 2(\sqrt{2} - 1). \tag{3}$$

The cosine formula (along with $b = c$) now gives

$$\cos A = \frac{2b^2 - a^2}{2b^2} = 1 - 2(\sqrt{2} - 1)^2$$
$$= 4\sqrt{2} - 5, \tag{4}$$

$$\therefore \quad 2\sin^2 \frac{1}{2}A = 1 - \cos A = 2(\sqrt{2} - 1)^2, \tag{5}$$

which yields $A = 2\sin^{-1}(\sqrt{2} - 1)$. □

We remark that using the work done above, we can similarly give a complete classification of triangles with three equalizers in terms of the inequalities to be satisfied by its sides. This classification is simpler than that given in [1]. We omit it as our concern here is triangles which have exactly two equalizers. Later (Theorem 4, in Part II) we shall revisit the problem.

In [1], the angle $2\sin^{-1}(\sqrt{2} - 1) \approx 48°56'23'' \approx 49°$ is denoted by A_0, and the author regards it as full of surprise and drama. This angle also comes up as an upper bound on the smallest angle in any triangle which has exactly two equalizers as we now show. (Later we shall see that this angle also plays a crucial role in the calculation of the probabilities with which a triangle at random has a given number of equalizers.)

Theorem 2. *In a triangle with exactly two equalizers, the smallest angle can be at most equal to* $A_0 = 2\sin^{-1}(\sqrt{2} - 1)$.

Proof. The case of an isosceles triangle follows from (ii) of the last theorem. (In fact, here the smallest angle actually equals A_0.) Now suppose that (i) holds, i.e. that $\triangle ABC$ is scalene and has two equalizers. Without loss of generality, suppose $a < b < c$. Then we claim that $A \leq A_0$ by showing that $\cos A \geq \cos A_0$. For such a triangle, we have $s^2 = 2bc$ and hence $(a + b + c)^2 = 8bc$, which implies $a = \sqrt{8bc} - b - c$. Putting this into the cosine formula and using the A.M.-G.M. inequality, we get:

$$\cos A = \frac{b^2 + c^2 - (\sqrt{8bc} - b - c)^2}{2bc}$$
$$= \frac{2\sqrt{8bc}(b + c) - 10bc}{2bc}$$
$$\geq \frac{4\sqrt{8}bc - 10bc}{2bc} = 4\sqrt{2} - 5 = \cos A_0,$$

which completes the proof. □

We now prove a result which is a sort of converse to this theorem.

Theorem 3. *Given any* α *with* $0 < \alpha \leq A_0$, *there exists a triangle which has exactly two equalizers and whose smallest angle is* α. *Moreover such a triangle is unique up to similarity.*

Proof. The case where $\alpha = A_0$ is already settled by Part (ii) of Theorem 1. So suppose $0 < \alpha < A_0$. We construct a scalene $\triangle ABC$ in which $\measuredangle A = \alpha$, a is the shortest side and $s^2 = 2bc$. Indeed, we let s be arbitrary. In view of $s^2 = 2bc$, the requirement that $\measuredangle A = \alpha$ is equivalent to

$$\cos\alpha = \frac{b^2 + c^2 - (2s - b - c)^2}{s^2}$$
$$= \frac{4(b + c)s - 5s^2}{s^2},$$

which reduces to

$$b + c = \frac{(5 + \cos\alpha)}{4}s. \tag{6}$$

We solve this simultaneously with $s^2 = 2bc$ and find that b and c are the roots of the quadratic equation

$$q^2 - \frac{(5 + \cos\alpha)s}{4}q + \frac{s^2}{2} = 0. \tag{7}$$

By a direct calculation, the roots of this quadratic are

$$\frac{(5 + \cos\alpha) \pm \sqrt{(5 + \cos\alpha)^2 - 32}}{8}s. \tag{8}$$

For the roots to be real and distinct, we must ensure that $(5 + \cos\alpha)^2 > 32$, which reduces to $\cos\alpha > 4\sqrt{2} - 5 = \cos A_0$. Since we have assumed that $\alpha < A_0$, this requirement is satisfied. Thus, the quadratic (7) has two distinct real roots.

We let b be the smaller root and c the larger one. Finally, we let

$$a = 2s - (b + c) = \left(2 - \frac{5 + \cos\alpha}{4}\right)s$$
$$= \frac{3 - \cos\alpha}{4}s, \qquad (9)$$

which is clearly positive. To ensure that a, b, c so defined form a scalene triangle with a as its shortest side, we must prove that $a < b$ and also that $a + b > c$. Recalling that b is given by the negative sign in (8), proving $a < b$ is equivalent to proving that

$$\frac{3 - \cos\alpha}{4} + \frac{\sqrt{(5 + \cos\alpha)^2 - 32}}{8} < \frac{5 + \cos\alpha}{8},$$

which simplifies to showing that

$$\sqrt{(5 + \cos\alpha)^2 - 32} < 3\ \cos\alpha - 1. \qquad (10)$$

The RHS is positive because $\cos\alpha > 4\sqrt{2} - 5$. Therefore, (10) can be proved by comparing the squares of both the sides. Finally, showing that $a + b > c$ is equivalent to showing that $a > c - b$ and reduces to proving that

$$\frac{3 - \cos\alpha}{4} > \frac{\sqrt{(5 + \cos\alpha)^2 - 32}}{4},$$

which, again, is proved by comparing the squares of both the sides.

Thus we have found a scalene $\triangle ABC$ with a as its shortest side, $\measuredangle A = \alpha$ and $s^2 = 2bc$. The only other such triangle would be one where b and c are interchanged. But these two triangles are congruent to each other. Since the ratios of the sides to each other are uniquely determined by α, the triangle is unique up to similarity. □

In the second part of this article, we shall show how to estimate the probability that a randomly chosen triangle has 1, 2 or 3 equalizers.

References

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