The Magical World of Infinities

Part 1

To see a World in a Grain of Sand
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand
And Eternity in an hour.

From *Auguries of Innocence* by **William Blake**

Features

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Introduction

This article and its sequel hope to take the reader on a whirlwind tour of the infinities! For those who have never encountered these ideas, be ready for your world to be turned upside down and your intuition to be shot to pieces. Don't worry, you're not the only one who may react violently to the ideas presented below. Reputed mathematicians like Poincaré and Kronecker reacted with horror. Georg Cantor, who discovered these ideas, was literally driven to insanity because of the hostility he received, especially at the hands of Kronecker. But before we get caught up in this story, let us begin our journey into the world of Infinity.

Ready? Do you have your seat belts fastened? Then let us start with the so-called Hilbert hotel (an idea introduced by the famous German mathematician David Hilbert). The Hilbert hotel has an infinite number of rooms, and all the rooms in the hotel are full. Now if a new guest arrives, can you accommodate her? Some of you may have figured it out already! Yes, move each guest to the adjacent room, that is: ask the guest in room 1 to move to room 2, the guest in room 2 to move to room 3, the guest in room 3 to move to room 4, and so on. As there are infinitely many rooms in

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the Hilbert hotel, you can accommodate everyone who has already checked in and you now have an empty room number 1, where you can accommodate the new guest!

Clever, you might say, but what if an infinite number of new guests arrive; can you accommodate all of them? The answer again is yes! This one needs a little more ingenuity. Here is what you do. Send the guest in room 1 to room 2, the guest in room 2 to room 4, the guest in room 3 to room 6, and so on. You have now managed to vacate room numbers 1, 3, 5 and so on. All the odd numbered rooms are vacant, and as there are infinitely many odd numbers, you can now accommodate the infinitely many new guests!

I am sure you think there is something fishy going on here. Now suppose an infinite number of new guests arrive in an infinite number of buses, can we accommodate all of them? The answer is still yes! It will be a while before we can show how to do it. In the meantime, let us put our common sense about infinite sets to some further testing!

1. *Which set has more elements, the set of all natural numbers or the set of even natural numbers?*
2. *Which set has more elements, the set of all natural numbers or the set of integers?*
3. *Which set has more elements, the set of all integers or the set of rational numbers?*
4. *In Figure 1, which circle has more points?*

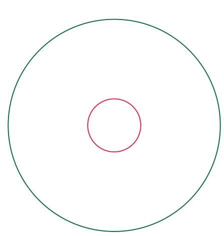

Figure 1

5. *In Figure 2, does the square have more points or does one of its edges?*

Figure 2

6. *In Figure 3, does the cube have more points or does one of its edges?*

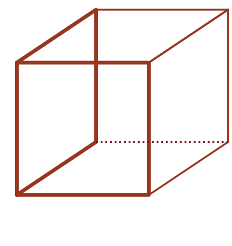

Figure 3

The answer to the first three questions is that in each case the two sets have the same number of elements, and the answer to the last three is that the two sets have the same number of points. Sounds crazy? Before we justify these answers, we need to lay some mathematical foundation to this madness!

One-to-one correspondence

You may have guessed that the basic issue here is: how do we compare infinite sets? How on earth can we say that one infinite set has more than, less than or equal number of elements as compared with another infinite set? A related question is, can we assign a 'value' to an infinite set?

Georg Cantor [1845–1918] was the first mathematician to dare to answer these questions. It is amazing that anyone could even conceive of the idea of counting infinite sets. What is even more startling is that the basic principle used is something even a young child can understand and uses all the time to count: the concept of **one-to-one correspondence**.

We say two sets A and B are in one-to-one correspondence (from now on, denoted by 1-1 correspondence) if there is a way of associating each element of A with a unique element of B, and similarly associating each element of B with a unique element of A. That is, every member of A has a unique partner in B, and every member of B has a unique partner in A. This principle is illustrated in Figure 4.

We see that there is a 1-1 correspondence between sets A and B. Moreover, we can also see that both sets A and B have the same number of elements.

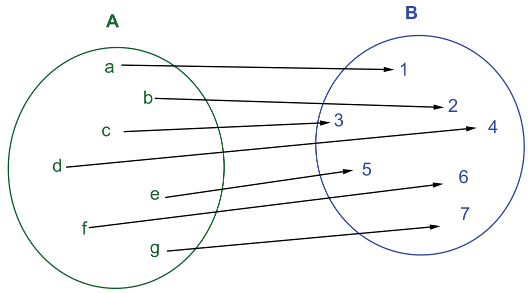

Figure 4

We often encounter this idea in daily life, for example if seats in a movie theatre are sold out, and if we know the theatre can house 472 seats, we can conclude (assuming there are no lap-tops!) that there are 472 people watching the movie.

Figure 5 shows an example of a function which is not a 1-1 correspondence. This is because more than one element in set C has the same partner in set D.

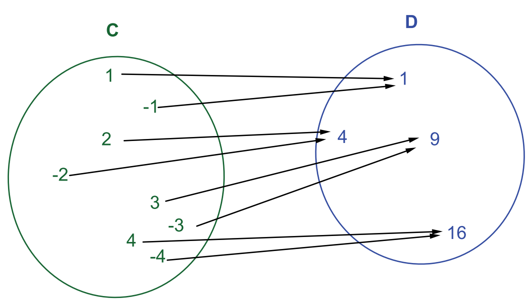

Figure 5

Cantor's brilliant insight was that the idea of 1-1 correspondence can be extended to infinite sets, and we can start comparing infinite sets and even count them. We say two sets A and B have the same number of elements if there is a 1-1 correspondence between them. In mathematical notation we write: $|A| = |B|$.

So in order to answer our six baffling questions, we need to create 1-1 correspondences between the pairs of sets in question. For example, if we want to compare the set of natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

with the set of even natural numbers

$$E = \{2, 4, 6, \ldots\},$$

we need to find a 1-1 correspondence between them. This one is easy! We send each natural number to its double; in function notation this would be:

$$f : \mathbb{N} \to E, \quad f(n) = 2n, \quad \text{where } n \in \mathbb{N}.$$

It is easy to see that for every natural number, we have associated an even number, and similarly for every even number we have associated a natural number. This is exactly the principle used in accommodating the infinitely many passengers who arrived at the Hilbert hotel.

Now because we have established a 1-1 correspondence, we can say that there are as many natural numbers as even numbers. What about odd natural numbers? We leave it as an exercise to show that there are as many natural numbers as odd natural numbers. Here we see the first among many strange aspects of infinities: a set and its subset can have the same size! Our intuition that the whole is always greater than its part goes out of the window!

What about the set of integers? Surely, since they also have the negative numbers, there should be more integers than natural numbers? We use a technique called *interlacing* (frequently used by Cantor) to demonstrate pictorially a 1-1 correspondence (see Figure 6). This is of course not a proof; would you be able to construct one? In fact since many of the proofs will be rather technical, we are going to use pictures to indicate how the proofs work. Interested readers can look up the references below for formal proofs.

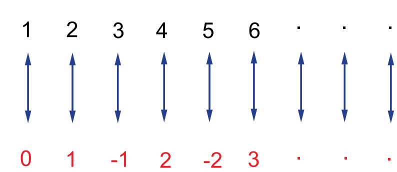

Figure 6

The question of whether the number of rational numbers equals the number of integers is a little more difficult, so we first tackle the question of the circles before we move to the other questions. Here is an easier question: In Figure 7, which line has more points?

Figure 7

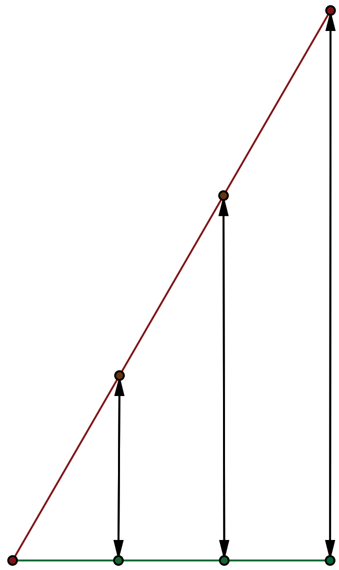

Figure 8

Consider the 1-1 correspondence shown in Figure 8. Readers will have to convince themselves that this can be done for any two pairs of lines with any length. The 1-1 correspondence clearly shows that the two lines have the same number of points. So, out goes the notion that longer lines have more points. (This raises an important question in a field of mathematics called Measure Theory: how does one define the length of a line?)

To answer the question about circles, once again we offer a pretty picture (Figure 9) and, a la Bhaskara, we say "behold"!

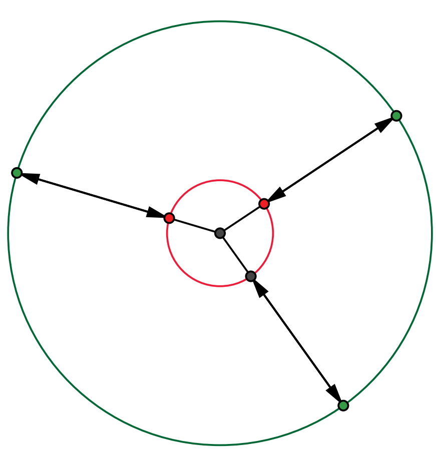

Figure 9

Before we go on to the question of the size of the set of rational numbers, let us look at one more comparison of sets of points. Consider all points in the interval $(-1, 1)$ on the real number line, and the real number line itself. Which has more points? By now, out of fatigue, you are probably saying 'the same'! Can we construct a 1-1 correspondence?

Here we can actually give the explicit 1-1 correspondence by the function

$$g : (-1, 1) :\to \mathbb{R}, \quad g(x) = \frac{2x}{1-x^2}.$$

Using algebra, you can establish that for every $t \in (-1, 1)$, $g(t)$ is a real number; for every real number s, $-1 < g(s) < 1$; and $g(t)$ does not repeat values, that is, if $g(a) = g(b)$ then $a = b$. This 1-1 correspondence shows that there are the same number of points in a line of finite length and in a line with infinite length. What is more, it shows that there are as many real numbers overall as there are between -1 and 1!

Let us look at the graph (Figure 10), where the 1-1 correspondence is self-evident (but not a proof!).

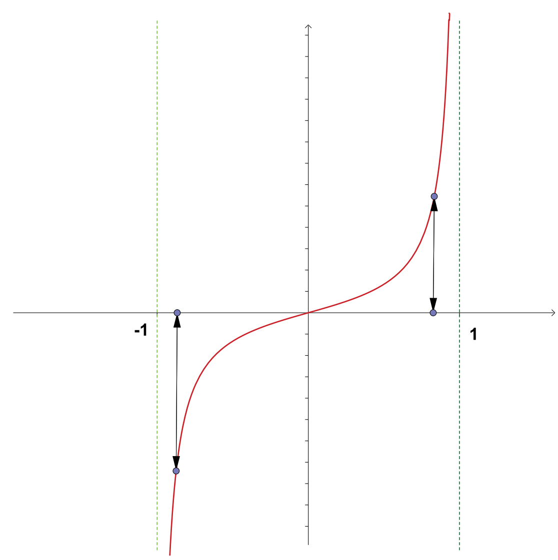

Figure 10

Counting the rational numbers and returning to the Hilbert hotel

Now we can take on the question about rational numbers. Here is where we encounter the genius of Cantor. He came up with the 1-1 correspondence shown in Figure 11 between the integers and the set of rational numbers. The explicit 1-1 correspondence is as shown in Figure 12.

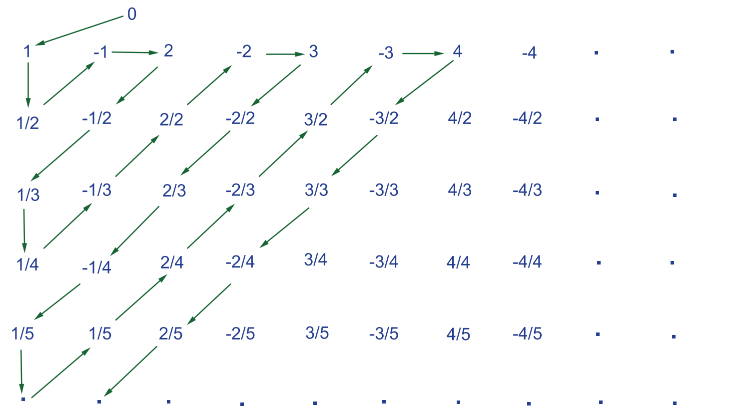

Figure 11

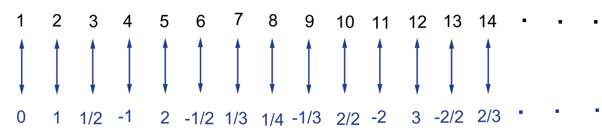

Figure 12

You might have noticed that the rational numbers repeat themselves, several times over. Does this cause a problem? Not really, because we have two options. One is to systematically skip any rational number that is repeated. Alternatively we use the following argument. Denote the collection of rational numbers with their repetitions by $\mathbb{Q}^*$. So we have shown that $|\mathbb{Q}^*| = |\mathbb{N}|$. Notice that $\mathbb{Q} \subset \mathbb{Q}^*$ and $\mathbb{N} \subset \mathbb{Q}$. Using a famous theorem (the Schroeder-Bernstein theorem), we can say $|\mathbb{Q}| \leq |\mathbb{N}| \leq |\mathbb{Q}|$ and hence $|\mathbb{Q}| = |\mathbb{N}|$.

Interestingly, the 1-1 correspondence shown above helps us solve the following question: if infinitely many buses arrive at the Hilbert hotel, each with infinitely many passengers, can we accommodate them? Of course we can. Let us look at Figure 13 which is a modification of Figure 11.

As you can see from Figure 13, we have arranged the infinitely many buses with their infinitely many passengers in rows and columns (so $B_2 4$ represents passenger 4 in bus 2). To accommodate all of them, once again shift each occupant as we did earlier to a room bearing double the number, and create infinitely many vacant rooms. We now use Cantor's method of criss-crossing to assign rooms to the passengers from the infinitely many buses, using the allocation shown in Figure 14.

How many infinities are there?

So far we seem to have encountered two kinds of sets. One is the set of rational numbers and its

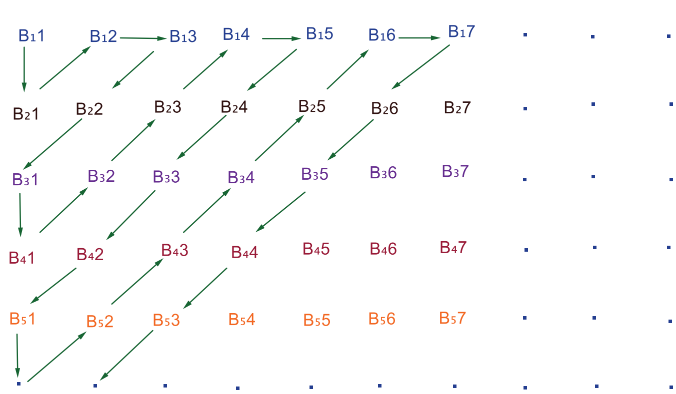

Figure 13

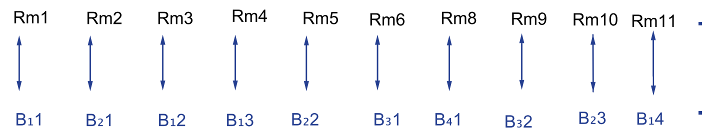

Figure 14

subsets, the integers, whole numbers and the natural numbers. We have seen that they all have the same number of elements in them. We also met points on curves and lines, and have been trying to compare them with each other and the real line. Here too we found that points on a line or a circle, no matter how long, have the same number of points. Now here's a question: how does the set of real numbers compare with the set of natural numbers? From our experience so far, we may be tempted to conclude that they are the same, especially when all our intuition about sizes of sets has been systematically destroyed! Once again Cantor pulls the rabbit out of the hat and shows that actually the set of real numbers has more elements than the set of natural numbers. In our language, he showed that no matter how clever you are, you cannot find a 1-1 correspondence between the set of natural numbers and the set of real numbers!

He went on to show much more. He denoted by the letter $\aleph_0$ the size (in mathematics: cardinality) of the set of natural numbers. The symbol $\aleph$ is the first letter in the Hebrew alphabet. He represented the cardinality of real numbers by **c**, the so-called 'continuum'. So, he showed

$$|\mathbb{N}| = \aleph_0 < \mathbf{c} = |\mathbb{R}|.$$

He assigned a symbol to represent the cardinality of any given infinite set (these symbols are called cardinal numbers), and showed that there exists a hierarchy among cardinal numbers, and in fact showed that $\aleph_0$ is the smallest cardinal number in this hierarchy.

In the second part of this article, we will discuss these ideas and return to the questions about the number of points in a square and one of its edges, and a cube and one of its edges!

Perhaps some of you will agree with Hilbert (talking about Cantor's work on infinities in 1926), when he says:

This appears to me to be the most admirable flower of the mathematical intellect and one of the highest achievements of purely rational human activity.

Maybe you will get hooked on to infinities the way I am!

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SHASHIDHAR JAGADEESHAN received his PhD from Syracuse in 1994. He is a teacher of mathematics with a belief that mathematics is a human endeavour; his interest lies in conveying the beauty of mathematics to students and looking for ways of creating environments where children enjoy learning. He is the author of *Math Alive!*, a resource book for teachers, and he has written articles in many education journals. He may be contacted at jshashidhar@gmail.com.