Of Art and Mathematics

Paradoxes:

True AND/OR False?

Part 1 of 2

feature

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This is the first sentence of this article.

Clearly the sentence above is true (not highly informative but true). Contrast this to the next sentence, below:

This is the first sentence of this article.

Now the second statement, though identical to the first, is clearly false.

Such sentences that speak about themselves are called *self-referential* sentences, because they are, in a way, looking at themselves in the mirror and describing themselves. Figure 1, is a design for the word "reference" so it looks the same when reflected in a mirror.

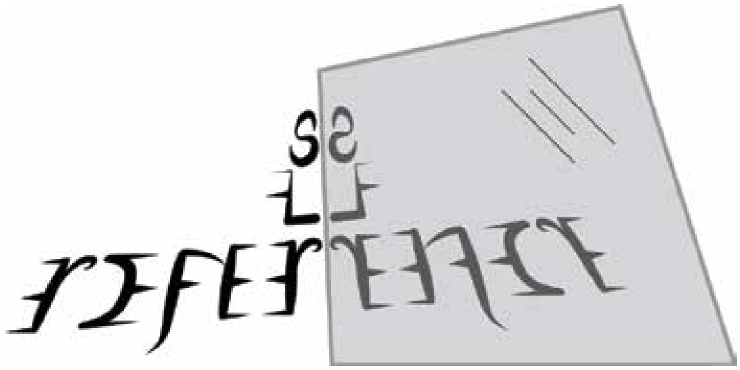

Figure 1. Self-reference looks in a mirror. The word "self-reference" is written in a manner that it looks the same when reflected in a mirror (a wall reflection).

Keywords: *Truth value, self-reference, paradox, axiom, theorem, consistency, circular argument, proof, Zeno, ourobouros, ambigram*

Figure 2. An ambigram for Paradox, the subject of this column

Such self-referential sentences sometimes lead to paradoxes, and paradoxes are the topic of this article. As usual we use the medium of ambigrams to communicate some of these paradoxical ideas (see Figure 2 for an ambigram of Paradox). And we produce some graphical paradoxes of our own for you to think about.

Mathematical Truth

To understand what self-referential statements have to do with mathematics we need to get a bit deeper into what mathematicians mean by the words *true* and *false*. A mathematical theory consists of a large number of statements. There are two special types of true statements in any mathematical theory–axioms and theorems.

For example, consider the development of plane geometry. We begin with certain axioms (such as: given a line and a point not on the line, there is exactly one line through that point parallel to the given line). Axioms are all considered to be true. Now by following the rules of logic, from Axioms one *proves* some other statements that are called theorems. If the proof is valid, we say the theorem is true. For example, a theorem is: The sum of three angles of a triangle is equal to two right angles. Each theorem is proved using the axioms, or the previously proved theorems. Figure 3 includes an ambigram of the word "axiom" that is then used over and over again to create an ambigram of the word "theorem."

Each statement in this theory is either true or false—it cannot be both, otherwise there will be a contradiction. And we will see shortly that contradictions are not allowed in mathematics.

Figure 3. Rotational ambigrams for the words "axiom" and "theorem"' – except that the word "theorem" is both an ambigram and constructed from the multiple axioms

Puzzle:

Can you decipher these strange squiggles below? Hint: There are two words related to this article

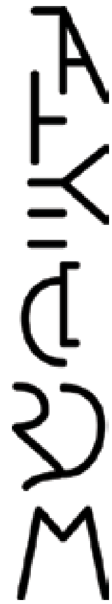

Figure 4. What do these squiggles mean?

In this theory, the axioms are taken to be true. However it is not necessary that the axioms are 'true' in every context. For example, the axioms of plane geometry are true in the idealized plane, but do not hold for the surface of the sphere, where 'lines' are simply *great circles*, which are formed by the intersection of the sphere with a plane passing through the center of the sphere. The equator, and lines of longitude are examples of great circles on a spherical globe. In this geometry, there is no line parallel to the given line from a point not on the line! This is because two great circles always meet. But surely the geometry of the sphere is equally "true" in the real world. (This kind of geometry, on the surface of the sphere, is called Riemannian Geometry).

What mathematical theories try to achieve is a consistency, where by consistency we mean: given the axioms and theorems proved within the theory (using the rules of logic), none of the statements contradict each other. Proofs are means to convince ourselves that the statements are "true" in the mathematical theory.

In developing a mathematical theory, one needs to be careful to avoid a circular proof. A circular proof is when the proof of a statement uses the statement itself! Figure 5 is a reflection chain ambigram of the word "proof" — a visual circular proof!

A circular argument can be difficult to find. Say in proving a statement P we use the truth of a

Figure 5. A visual representation of a circular proof! This design reads the same both at the front (as in red) or at the back — or even when read in a mirror.

statement Q. But the proof of the statement Q involves the statement P. A good example of circular reasoning is in the book Catch 22,

> "You mean there's a catch?"
>
> "Sure there's a catch", Doc Daneeka replied. "Catch-22. Anyone who wants to get out of combat duty isn't really crazy."
>
> There was only one catch and that was Catch-22, which specified that a concern for one's own safety in the face of dangers that were real and immediate was the process of a rational mind. Orr was crazy and could be grounded. All he had to do was ask; and as soon as he did, he would no longer be crazy and would have to fly more missions. Orr would be crazy to fly more missions and sane if he didn't, but if he was sane, he had to fly them. If he flew them, he was crazy and didn't have to; but if he didn't want to, he was sane and had to. Yossarian was moved very deeply by the absolute simplicity of this clause of Catch-22 and let out a respectful whistle.
>
> "That's some catch, that Catch-22," he observed.
>
> "It's the best there is," Doc Daneeka agreed.

Or in the character Tippler in the Little Prince who says he drinks so that he may forget that he is ashamed of drinking! As the little prince says,

> "The grown-ups are certainly very, very odd."

In mathematics circular proofs show up when something that is assumed is then used to prove the same thing. For instance here is a circular proof of the Pythagorean theorem.

> Let ΔABC be a right triangle with sides a, b, c. As usual, let c be the hypotenuse, the side opposite the right angle C. We know that $\sin B = b/c$ and $\cos B = a/c$.
>
> Now using the elementary trigonometric identity $\cos^2 B + \sin^2 B = 1$, we find that $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, or $a^2 + b^2 = c^2$, as required.

The only problem with this proof is that it presupposes the Pythagorean theorem—the very theorem that it sets out to establish.The proof of $\cos^2 B + \sin^2 B = 1$ relies on the Pythagorean Theorem! This is a good example of a vicious circle (see the design for *ouroboros*, Figure 6, for another, more lethal, variant of a vicious circle!).

Why do Mathematicians not allow any contradictions in the theories they build?

Mathematicians avoid contradictions because they can completely destroy the entire theory. This is because of a theorem of logic: *a false proposition implies any proposition*. Given that a false statement implies *any* statement, there is not much point in having a theory that has false

Figure 6. A chain rotation ambigram for the word "ouroboros" representing the idea of a snake eating its own tail. The idea of the ouroboros has recurred throughout history – such as the image in the middle, which is from a late medieval alchemical manuscript (courtesy Wikimedia Commons).

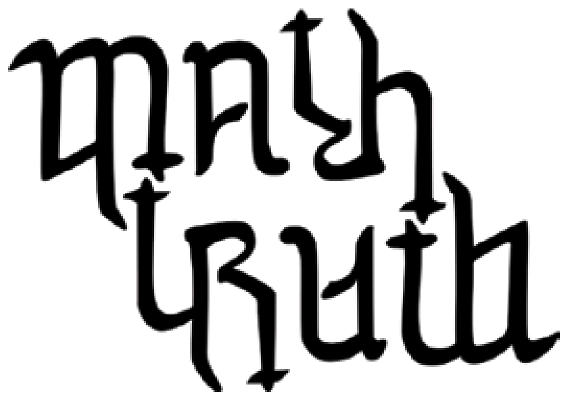

Figure 7. An ambigram about the relationship of math to truth

statements. For instance, on the one hand, we can prove a statement such as: There are an infinite number of prime numbers (as Euclid did over 2000 years ago). However, if even *one* false statement creeps into our mathematical universe, we can also prove that: There are only finitely many prime numbers! Or that there are exactly 317 prime numbers. Or that there are no prime numbers! Or that prime numbers are made of sweet buttermilk!

An example of a 'Proof' using a false proposition is this famous (probably apocryphal)story about the philosopher and mathematician Bertrand Russell (as retold by Raymond Smullyan in his classic book *What is the name of this book?).* Russell once told a dinner audience that "a false proposition implies any proposition." He was challenged to show that if $2+2=5$ (clearly a false statement) then he could prove that he (Russell) is the Pope. Russell then responded as follows:

> Given that $2+2=5$. Subtract 3 from both sides to get $1=2$. Now consider the following statements: The Pope and I are two. But $2=1$. So the Pope and I are one. Thus I am the Pope!

Note that starting from a false statement we end up with a nonsensical statement that "Russell is the Pope'. Thus something is wrong with the argument.

Mathematicians avoid contradictions like the plague (even more than writers avoid clichés). This is the reason why we insist on proofs in mathematics—to convince ourselves that all the statements are true. Figure 6 is a design where "math" rotates to read the word "truth."

Sometimes contradictions lead to paradoxes (or apparent paradoxes). Paradoxes are contradictory statements and have to be false. But since false statements are not allowed, there has to be some flaw in the reasoning. Resolving these paradoxes helps us understand the flaws in our reasoning. And more importantly, thinking about these paradoxical situations is fun.

Before we get into some serious self-contradictory paradoxes here is one that goes back a while – and one that turns out not to be a paradox if addressed with the right mathematical tools.

Zeno's Paradox

Zeno's paradoxes are about the impossibility of motion. A simple example is as follows. Suppose you have to go from a point A to a point B, which are 1 km distant from each other. Then first you have to reach halfway, a distance of half a km away. Then you have to go from mid point of AB (say A_1) to B. Again you have to first go half the distance A_1B which is one-fourth of a km. Next we have to go half the remaining distance, that is, one-eighth of a km. Going on in this fashion, Zeno asserted that we can never reach B. In other words, it is impossible to go from A to B. Thus Zeno showed by this argument that motion is impossible!

What is wrong with Zeno's argument? Zeno's paradoxes forced philosophers and mathematicians to think of the *continuum* and concepts such as infinite series. In our example above, we find that

$$1=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

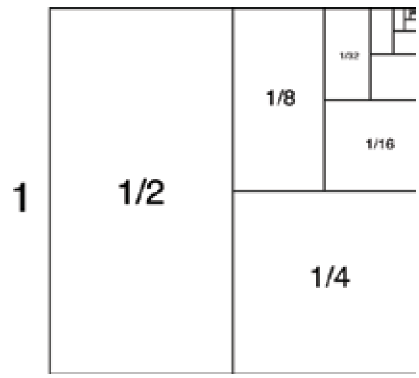

Figure 8. A 'proof by pictures' of the sum of the geometric series and how an infinite number of additions can lead to a finite sum

Figure 9. A visual Zeno Paradox, where "Zeno" gradually transforms to "Zero" – where the letter "n" changes step by step to the letter "r." Is Zeno ever Zero?

which follows from the formula for the sum of the geometric series. Figure 8 shows a "proof by pictures" of this series. We can use the concept of infinite series to resolve Zeno's paradox, by noting that the sum of an infinite number of additions can be a finite quantity.

Figure 9 shows an ambigrammatic approach to Zeno's paradox; here the word Zeno tends to Zero!

In the Geometric Series, the infinite sum is a finite quantity. The ambigram of Figure 10 is about the word "Finite" written in such a manner that it becomes the symbol for infinity!

Figure 10. Finite reflection in a circle. The word finite repeats in a circle – and is also reflected in a mirror. Taken together the main image and its reflection from the symbol for infinity.

In conclusion

With this we come to the end of our first part of our reconnaissance of the domain of paradoxes in mathematics. There is a lot more to come…but for that you will have to wait for part 2 of this article.

So with that, we should let you know that though it may seem that way, *this* sentence is surely not the last word on the topic. This is. No. This. Word.

Answer to puzzle: If you place a mirror vertically along the middle of the squiggles you will see two words – Axiom and Theorem (as follows).

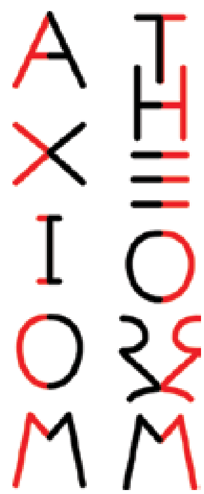

Figure 11. Solution to Puzzle 1

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Over the years, they have shared their love of art, mathematics, bad jokes, puns, nonsense verse and other forms of deep-play with all and sundry. Their talents however, have never truly been appreciated by their family and friends.

Each of the ambigrams presented in this article is an original design created by Punya with mathematical input from Gaurav (except when mentioned otherwise). Please contact Punya if you want to use any of these designs in your own work.

To you, dear reader, we have a simple request. Do share your thoughts, comments, math poems, or any bad jokes you have made with the authors. Punya can be reached at punya@msu.edu or through his website at http://punyamishra.com and Gaurav can be reached at bhatnagarg@gmail.com and his website at http://gbhatnagar.com/.