feature

How do we explain their success?

Understanding the Formulas

High-school math to the rescue …

In the accompanying article Approximating Square Roots and Cube Roots, *the author Ali Ibrahim Hussen has proposed easy-to-use formulas for finding approximate values of the square root and cube root of an arbitrary positive number n. The formulas are found to give fairly satisfactory results, as measured by the low percentage error. In this article we explain mathematically why this is so.*

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Square root

Both methods start by writing n as $a \times b$, where a and b are close to each other. Now we have the following:

$$\sqrt{n} = \sqrt{ab} = \sqrt{a \times a \times \frac{b}{a}} = a\sqrt{\frac{b}{a}}.$$

Let $b/a = 1 + x$. The fact that "b is close to a" translates to: "$1 + x$ is close to 1", i.e., "x is close to 0". This is generally written as: $x \approx 0$. (Another way of writing it is: $x \ll 1$.) With this notation we have: $\sqrt{n} = a\sqrt{1+x}$.

Accordingly, it is sufficient if we study what the two algorithms yield for the value of $\sqrt{1+x}$.

Keywords: *Square root, cube root, estimation, accuracy, geometric mean, harmonic mean, inequality, square, cube*

First approximation to the square root. We write $1+x = 1 \times (1+x)$. What we need is the geometric mean (GM) of 1 and $1+x$, but we approximate the GM by the arithmetic mean (AM), i.e., by $(1+1+x)/2 = 1+x/2$. Hence the first approximation to the desired square root is:

$$\sqrt{1+x} \approx 1 + \frac{x}{2}. \qquad (1)$$

In analytic terms this approximation is well known and easy to understand, for it is the *linear approximation* to the function $\sqrt{1+x}$ as given by the binomial theorem. Recall the statement of the binomial theorem for exponents n other than positive integers:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \cdots,$$

where $|x| < 1$ for convergence of the infinite series on the right side. (We must have such a condition, because if n is not a positive integer then the series does not terminate. If however n is a positive integer, the expression on the right side is simply a polynomial in x, of degree n, so the question of convergence does not arise, and the statement is then an identity, valid for all x.)

For the particular case $n = 1/2$, we get the series for the square root of $1+x$:

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots. \qquad (2)$$

If x is close to 0, then we may drop all terms with degree 2 or more, on the ground that they are small and do not make much of a difference to the total. The resulting formula, $1+x/2$, is called the "linear approximation" to $\sqrt{1+x}$. Thus the first approximation presented in Hussen's article is equivalent to the linear approximation. Table 1 shows how well this formula does, and Figure 1 displays the same relationship in graphical terms.

We see that if x is close to 0, the linear approximation gives good results. Obviously, better than the linear approximation is the quadratic approximation, $1+x/2-x^2/8$, and still better is the cubic approximation. But we leave the testing of these formulas to the reader.

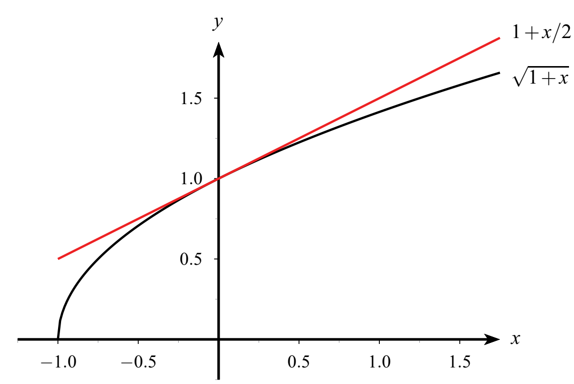


Figure 1. Graphical study of the approximation $\sqrt{1+x} \approx 1 + x/2$

Second approximation to the square root. As earlier we write $1+x = 1 \times (1+x)$. Now we approximate the GM of 1 and $1+x$ (which is what is required) by the average of the AM and the harmonic mean (HM) of 1 and $1+x$. We have:

$$\text{Arithmetic mean (AM)} = \frac{2+x}{2},$$

$$\text{Harmonic mean (HM)} = \frac{2(1+x)}{1+(1+x)} = \frac{2(1+x)}{2+x},$$

$$\text{Average of AM and HM} = \frac{1}{2}\left(\frac{2+x}{2} + \frac{2+2x}{2+x}\right).$$

The last expression when simplified yields a fresh estimate for the square root of $1+x$:

$$\frac{1+x+x^2/8}{1+x/2}. \qquad (3)$$

x	1	0.2	0.1	0.01	0.001
$1+x/2$	1.5	1.1	1.05	1.005	1.0005
$\sqrt{1+x}$	1.414	1.0954	1.0488	1.004987	1.00049987
% error	6.065	0.4158	0.1135	0.00124	0.0000126

Table 1. Error study of the first approximation

x	1	0.2	0.1	0.01	0.001
$(\text{AM} + \text{HM})/2$	1.41667	1.09545	1.04881	1.00499	1.0005
$\sqrt{1+x}$	1.414	1.0954	1.0488	1.004987	1.00049987
% error	0.173	0.00086	0.000064	7.66×10^{-9}	7.77×10^{-13}

Table 2. Error study of the second approximation

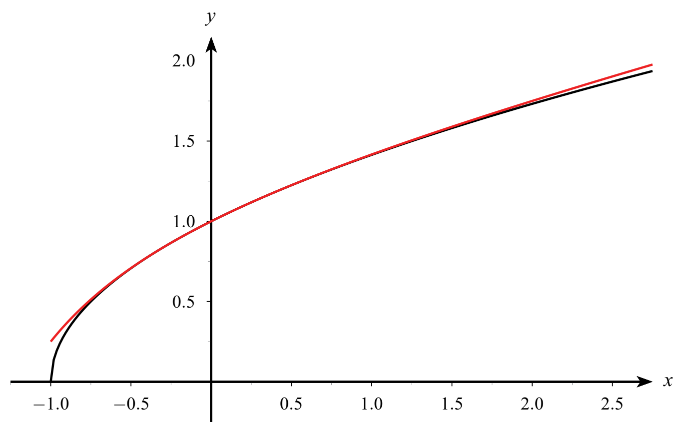


Figure 2. Graphical study of the approximation $\sqrt{1+x} \approx 1/2\,[1 + x/2 + 2(1+x)/(2+x)]$

Table 2 shows how well the new formula does, while Figure 2 displays the same relationship in graphical terms. Note the closeness of the two graphs even for values of x that one would not consider 'small'. We see that the second approximation does significantly better than the first formula — far better than one would ever have expected.

Insight into why this formula does so well comes when we examine the relevant power series expansion, which we get using the binomial theorem. We have:

$$\frac{1 + x + x^2/8}{1 + x/2} = \left(1 + x + \frac{x^2}{8}\right)\left(1 + \frac{x}{2}\right)^{-1}$$

$$= \left(1 + x + \frac{x^2}{8}\right)\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16} - \cdots\right)$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{x^4}{32} + \cdots$$

$$\text{(on multiplying out, term by term).} \qquad (4)$$

Compare this with:

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots.$$

We see that the two series coincide right up to the x^3 term (and even their x^4 coefficients are quite close to each other). No wonder that the two formulas agree so closely.

Cube root

Following the discussion in the previous section, it is sufficient if we focus on estimating the cube root of $1 + x$ where $x \approx 0$. The proposed method starts by expressing $1 + x$ as a product of three numbers close to each other. We shall write $n = 1 \times 1 \times (1 + x)$, i.e., our three numbers are $1, 1, 1 + x$. Then the task of computing $(1 + x)^{1/3}$ is equivalent to: Compute the geometric mean (GM) of $1, 1, 1 + x$.

As we did with the square root, we start with the arithmetic mean of these three numbers; we get $d = 1 + x/3$. Now we try to improve this estimate by subtracting some quantity h from d such that $(d - h)^3 = 1 + x$. Write $e = d - h = 1 + x/3 - h$, so e is going to be our new estimate for the desired cube root. Note that $h = 1 + x/3 - e$. We argue as follows:

$$1 + x = d^3 - 3d^2h + 3dh^2 - h^3,$$

$$\therefore\ d^3 - 1 - x \approx 3d^2h - 3dh^2$$

$$\text{(we drop the } h^3 \text{ term since } h \approx 0\text{)},$$

$$\therefore\ d^3 - 1 - x \approx 3dh(d-h),$$
$$\therefore\ \frac{d^3-1-x}{3d} \approx h(d-h),$$
$$\therefore\ \frac{(1+x/3)^3-1-x}{3+x} \approx e\left(1+\frac{x}{3}-e\right). \qquad (5)$$

The equation in the last line must be solved for e to give us our estimate for $(1+x)^{1/3}$. As the equation is quadratic, the manipulations can be done using known formulas. The algebra is tedious, but after some work (and a fair bit of help from a reliable computer algebra system like *Mathematica*!) we get the following result:

$$e = \frac{\left(\begin{array}{l}27 + 18x + 3x^2 \\ \quad + \sqrt{3\,(243 + 324x + 54x^2 - 12x^3 - x^4)}\end{array}\right)}{18(3+x)}. \qquad (6)$$

Now we must expand this expression into an infinite series, and then we need to compare the result with the binomial expansion of $(1+x)^{1/3}$. The binomial expansion for the cube root is easily found, using the binomial theorem:

$$(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \frac{154x^6}{6561} + \frac{374x^7}{19683} + \cdots. \qquad (7)$$

Generating the corresponding series for the expression e involves a lot more work, and this time we very definitely need the services of *Mathematica* to simplify the expressions involved. Here is what we get:

$$e = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \frac{157x^6}{6561} + \frac{395x^7}{19683} + \cdots. \qquad (8)$$

The two series agree all the way till the fifth degree term, and even their sixth and seventh degree terms do not differ by very much. Well! Now we understand why the method gives such astonishingly good results.

Another algorithm for cube root

Following the success of the second algorithm for square root, we propose something similar for the cube root. It does not do nearly as well as Hussen's algorithm presented above. On the other hand, that algorithm requires us to compute a square root at one stage, and this diminishes its attractiveness somewhat. It would be nicer if we could get a cube root estimate without any square root computation! — i.e., by sticking only to rational operations. We present one such possibility here. It is about as simple-minded as any algorithm can be, yet it does quite well.

Simple-minded algorithm for cube root

Step 1: Write n as $a \times b \times c$ where a, b and c are close to each other.

Step 2: Compute the AM and the HM of a, b and c.

Step 3: Compute the average of the AM and HM computed in Step 2. This is our estimate for $n^{1/3}$.

For example, take $n = 10$. Write 10 as $10 = 2 \times 2 \times 5/2$. Then we have:

$$\text{AM} = \frac{2+2+5/2}{3} = \frac{13}{6},$$

and:

$$\text{HM} = \frac{3}{1/2+1/2+2/5} = \frac{3}{7/5} = \frac{15}{7}.$$

Hence our estimate for the cube root of 10 is:

$$\frac{1}{2}\left(\frac{13}{6}+\frac{15}{7}\right) = \frac{181}{84} \approx 2.1548.$$

Here is the actual value: $10^{1/3} = 2.1544$. That seems close enough given how little we worked for it!

To study it analytically, we check the series expansion it gives for the cube root of $1+x$, after writing it as $1 \times 1 \times (1+x)$. We have, for the numbers $1, 1, 1+x$:

$$\text{Arithmetic mean (AM)} = \frac{3+x}{3},$$

$$\text{Harmonic mean (HM)} = \frac{3}{1/1+1/1+1/(1+x)} = \frac{3(1+x)}{3+2x},$$

$$\text{Average of AM and HM} = \frac{9+9x+x^2}{9+6x}.$$

So our estimate for the cube root of $1+x$ is

$$\frac{9+9x+x^2}{9+6x} = \frac{1+x+x^2/9}{1+2x/3}. \qquad (9)$$

The series expansion for the last expression is:

$$\begin{aligned}\frac{1+x+x^2/9}{1+2x/3} &= \left(1+x+\frac{x^2}{9}\right) \\ &\quad \left(1-\frac{2x}{3}+\frac{4x^2}{9}-\frac{8x^3}{27}+\cdots\right) \\ &= 1+\frac{x}{3}-\frac{x^2}{9}+\frac{2x^3}{27}-\frac{4x^4}{81}+\cdots. \end{aligned} \qquad (10)$$

Comparing this with the binomial expansion of $(1+x)^{1/3}$,

$$(1+x)^{1/3} = 1+\frac{x}{3}-\frac{x^2}{9}+\frac{5x^3}{81}-\frac{10x^4}{243}+\cdots, \qquad (11)$$

we see that the two series agree only till the x^2 term. The coefficients of the x^3 terms are close to each other but not the same, and likewise for the x^4 terms. This explains why the results of this recipe are only moderately good.

An algorithm for higher order roots

The idea proposed in the last section may be extended easily to k^{th} roots where k is an arbitrary positive integer.

Simple-minded algorithm for k^{th} root of $1+x$

Step 1: Write $1+x$ as $\underbrace{1\times 1\times\cdots\times 1}_{(k-1)}\times(1+x)$.

Step 2: Compute the AM and the HM of the list $\underbrace{1,1,\ldots,1}_{(k-1)}, 1+x$:

$$\text{AM} = \frac{1+1+\cdots+1+(1+x)}{k} = \frac{k+x}{k},$$

$$\begin{aligned}\text{HM} &= \frac{k}{1/1+1/1+\cdots+1/1+1/(1+x)} \\ &= \frac{k(1+x)}{k+(k-1)x}.\end{aligned}$$

Step 3: Compute the average of the AM and HM computed in Step 2. This is our estimate for $(1+x)^{1/k}$:

$$(1+x)^{1/k} \approx \frac{2k^2+2k^2x+(k-1)x^2}{2k^2+2k(k-1)x}. \qquad (12)$$

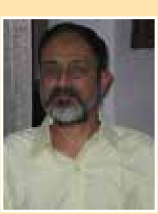
For example, take $x=1/4$ and $k=5$. That is, we want the fifth root of $5/4$, i.e., $(1.25)^{1/5}$. The formula yields:

$$\frac{50+50/4+1/4}{50+10} = \frac{251}{240} \approx 1.0458.$$

For comparison here is the actual value: $1.25^{1/5} = 1.0456$.

Closing remarks

Finding good rational approximations for irrational difficult-to-compute numbers is a theme that goes far back in history, and over the ages many remarkable algorithms have been devised. The venerable "long-division square root algorithm" is part of this tradition, as is a remarkable formula found by Bhāskara I to compute values of the sine function. (This will be the subject of a future article.) What is satisfying and of interest is that we are able to account for the good behaviour of the various algorithms in the accompanying article by Hussen. It reaffirms our faith in the subject!



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