feature

Is a picture worth ...

On Proofs Without Words

...a thousand words?

A "proof without words" sounds like a contradiction in terms! How can you prove something if you are not permitted the use of any words? In spite of the seeming absurdity of the idea, the notion of a proof without words — generally shortened to PWW — has acquired great popularity in mathematics in recent decades, and every now and then we come across new, elegant PWWs for old, familiar propositions. In this short article the seemingly contradictory nature of a PWW is discussed, and some examples of PWWs are presented.

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Introductory remarks. In recent decades there has been much interest in "proofs without words" (PWWs for short) which, as the Math Wolfram source [6] compactly puts it, are proofs "...only based on visual elements, without any comments." PWWs today form a whole new genre of proofs. Two of the magazines published by the Mathematical Association of America (MAA) — *College Journal of Mathematics* and *Mathematics Magazine* — regularly publish original PWWs sent in by readers. Two PWW anthologies ([2] and [3]) have appeared in book form (they contain nothing but PWWs), largely culled from the magazines mentioned, and there are a few web pages too ([1], [4], [5]) which have nice collections of their own.

What exactly is a PWW? The Wikipedia source [5] has the following to say: "In mathematics, a proof without words is a proof of an identity or mathematical statement which can be

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demonstrated as self-evident by a diagram without any accompanying explanatory text. Such proofs can be considered more elegant than more formal and mathematically rigorous proofs due to their self-evident nature. When the diagram demonstrates a particular case of a general statement, to be a proof, it must be generalizable."

But can there really be such a thing as a proof without words, or is it a self-contradictory notion? Consider what a mathematical proof is supposed to be: an argument written out in clear, understandable language, starting with a given set of propositions, justifying each step (generally done by referring to a proposition that has already been proved), and culminating in the proposition to be proved. Thus, every step is made *formal* and *explicit*.

At least that's the way it is supposed to be. In practice there are lots of statements which are not justified, on the ground that they are "obvious". Look through any proof and sooner or later you will meet these phrases: "it should be clear that …", "now obviously …", "it is quite obvious that …"; or phrases similar to these in meaning and intent. Perhaps that's the way it has to be; how can one possibly justify every single statement? The following fact is noteworthy: when a published proof has been found to be incorrect, the error almost always is found to lie concealed in such phrases. What seems obvious while writing the proof is not only *not* obvious, it may actually be false!

So where does that leave us with regard to PWWs? From the above comments it follows that in the strict formal sense of the word, a PWW is *not* a proof. Rather, it is a *suggestion of a proof*; it is an *outline of a proof*. Expressed another way, it is *proof cast in a poetic metaphor*. In a PWW, there is a kind of non-verbal communication going on between the author and the reader, and in that communication lie enough hints for the entire proof to be reconstructed. This means that a PWW depends on a shared culture of mathematics for its meaning: there is a common language being used by the author and the reader, a common lexicon or vocabulary. Without such a shared base, the PWW would be incomprehensible.

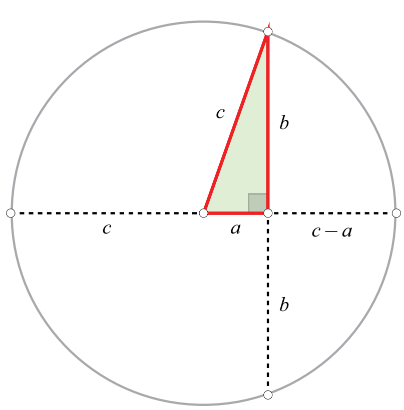
Viewed against the backdrop of such comments, there seems no substantive reason for not regarding a PWW as a proof. Accordingly, we shall accept the descriptions given above by [5] and [6].

We give below a few PWWs which are of particular elegance, along with their sources (when available). We hope that they will convince any sceptical reader of the value and worth of the PWW as a valid genre of proof.

A gallery of proofs without words

The theorem of Pythagoras. We start (naturally enough) with the venerable theorem of Pythagoras. We have featured the famous twelfth century "Behold!" proof of this theorem (due to Bhāskara II) in an earlier issue of *At Right Angles*, so we do not repeat it here. Instead we present a proof based on circle properties — specifically, the intersecting chords theorem, also called the 'crossed chords theorem'. It has been adapted from [2], page 8. See Figure 1.

The cosine rule. A small adaptation of the PWW for the theorem of Pythagoras yields a PWW for the cosine rule; it too draws from the intersecting



$$(c+a)\times(c-a)=b\times b$$
$$\therefore\ c^2-a^2=b^2$$
$$\therefore\ a^2+b^2=c^2$$

Figure 1. PWW for the theorem of Pythagoras

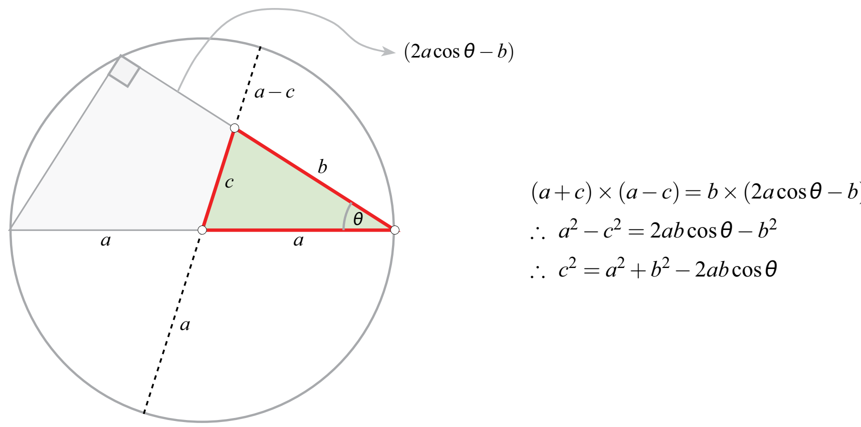


Figure 2. PWW for the cosine rule

chords theorem for its inspiration. This PWW is from [2], page 32. See Figure 2.

The tangent of 15 degrees. What is the value of tan 15°? Figure 3 (if we have drawn it properly, and if this PWW is as effective as it claims to be) should reveal the answer! Namely, it should convince you that $\tan 15^\circ = 2 - \sqrt{3}$. Please study the figure carefully, and let us know if it has persuaded you to agree with the statement.

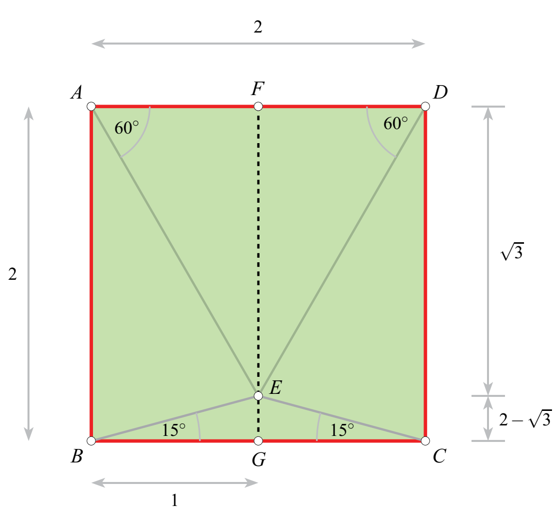


Figure 3. PWW to show that $\tan 15^\circ = 2 - \sqrt{3}$

Triangular number identity. The triangular numbers T_n ('T-numbers') are defined to be the partial sums of the sequence of natural numbers $1, 2, 3, 4, \ldots$ (so they are the numbers 1, $1 + 2 = 3$, $1 + 2 + 3 = 6$, $1 + 2 + 3 + 4 = 10$, ...). They are generated by the formula

$$T_n = \frac{n(n+1)}{2}.$$

The T-numbers exhibit a large number of identities which are closely intertwined with properties of the square numbers. Among the simplest and most charming of these are: (a) The sum of two consecutive T-numbers is a perfect square. (b) If you multiply a T-number by 8 and add 1 to the result, you get a perfect square. There are nice PWWs for both these properties which we leave you to find. For now we present a PWW for a less obvious and much less well-known result, taken from [2], page 104. Here is the result itself:

$$3T_n + T_{n-1} = T_{2n}.$$

The PWW is depicted in Figure 4. Note that the figure has been drawn for the specific case $n = 5$, so it only shows that $3T_5 + T_4 = T_{10}$. But it generalizes in a fairly obvious way to show that $3T_n + T_{n-1} = T_{2n}$ for all positive integers n.

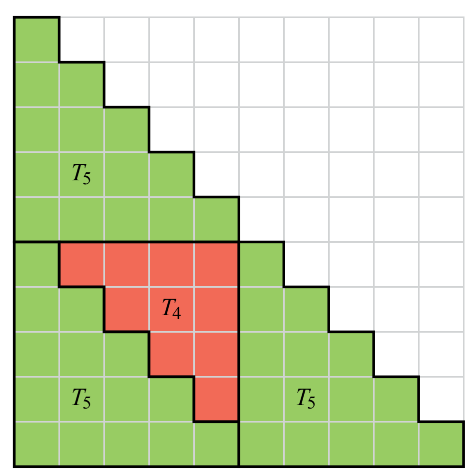
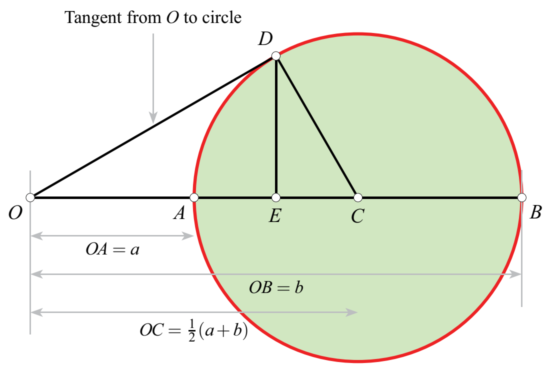


Figure 4. PWW to show that $3T_5 + T_4 = T_{10}$

In this PWW we see a theme which is very common in PWWs for number relations: *the PWW is shown only for a specific number*. But the way it is drawn gives a clear suggestion how it can be drawn for any number. *The passage to generalization is implicit in the way the figure is drawn.*



- $OD^2 = OA \times OB$
- $OD = \sqrt{ab}$
- $\frac{OE}{OD} = \frac{OD}{OC} \quad (= \cos\measuredangle DOE)$
- $OE = \frac{OD^2}{OC} = \frac{2ab}{a+b}$
- $OE =$ harmonic mean of a, b
- $OD =$ geometric mean of a, b
- $OC =$ arithmetic mean of a, b
- $OE < OD < OC$

Figure 5. PWW for the AM-GM-HM inequality

The AM-GM-HM inequality. We close this anthology with a PWW for the AM-GM-HM inequality, which plays a significant role in the article by Hussen elsewhere in this issue, *Simple Formulas for Square Roots*. The “AM-GM-HM inequality” is the statement that for any two positive numbers a and b, we have $\text{AM} \geq \text{GM} \geq \text{HM}$, where AM, GM and HM denote the arithmetic mean, the geometric mean and the harmonic mean respectively of a and b:

$$\text{AM} = \frac{a+b}{2},$$

$$\text{GM} = \sqrt{ab},$$

$$\text{HM} = \frac{2ab}{a+b}.$$

Moreover, equality holds precisely when $a = b$. The configuration depicted in Figure 5 demonstrates the property in a beautiful and succinct manner. (It has been drawn assuming that $a < b$.) Strictly speaking, this is *not* a PWW, as the derivations have been shown at the right side! But we have included it here as the figure puts the inequality into the framework of geometry in such a nice way.

References

[1] Bogomolny, A. Proofs Without Words. From “Interactive Mathematics Miscellany and Puzzles”, http://www.cut-the-knot.org/ctk/pww.shtml, Accessed 18 November 2014

[2] Nelsen, Roger B. *Proofs Without Words: Exercises in Visual Thinking*. MAA

[3] Nelsen, Roger B. *Proofs Without Words II: More Exercises in Visual Thinking*. MAA

[4] Proofs without words. http://mathoverflow.net/questions/8846/proofs-without-words

[5] https://en.wikipedia.org/wiki/Proof_without_words

[6] http://mathworld.wolfram.com/ProofwithoutWords.html

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