problem corner

# Problem for the Senior School

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## Problems for Solution

**Problem III-2-S.1**
Let $n$ be a positive integer not divisible by 2 nor by 5. Prove that there exists a positive integer $k$, depending on $n$, such that the number 111 ... 1, where the digit 1 is repeated $k$ times, is divisible by $n$.

**Problem III-2-S.2**
Let $\mathbb{R}$ be the set of all real numbers, and let $b$ be a real number, $b \neq \pm 1$. Determine a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) + bf(-x) = x + b$, for all $x \in \mathbb{R}$.

**Problem III-2-S.3**
Determine all three-digit numbers $N$ such that: (i) $N$ is divisible by 11, (ii) $N/11$ is equal to the sum of the squares of the digits of $N$. (This problem appeared in the International Mathematical Olympiad 1960.)

**Problem III-2-S.4**
You are given a right circular conical vessel of height $H$. First, it is filled with water to a depth $h_1 < H$ with the apex downwards. Then it is turned upside down and it is observed that water level is at a height $h_2$ from the base. Prove that

$$h_1^3 + (H - h_2)^3 = H^3.$$

Can $h_1$, $h_2$ and $H$ all be positive integers?

**Problem III-2-S.5**
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined as follows: $a_1 = 3, a_2 = 5$ and:

$$a_{n+1} = |a_n - a_{n-1}|, \quad \text{for all } n \geq 2.$$

Prove that $a_k^2 + a_{k+1}^2 = 1$ for infinitely many positive integers $k$.

## Solutions of problems in Issue-III-1

**Solution to problem III-1-S.1** *Let $f(x) = ax^2 + bx + c$, where $a, b, c$ are positive integers. Show that there exists an integer $m$ such that $f(m)$ is a composite number.*

We shall prove this by actually exhibiting such an integer $m$. Let $f(1) = n$; then $n = a + b + c$. Now consider the value of $f(n + 1)$:

$$f(n+1) = a(n+1)^2 + b(n+1) + c$$
$$= an^2 + 2an + bn + (a+b+c)$$
$$= an^2 + 2an + bn + n = n(an + 2a + b + 1).$$

Since $a, b, c$ are positive integers, $n > 1$ and $an + 2a + b + 1 > 1$. So $f(n+1)$ is a product of two integers both of which exceed 1. Therefore $f(n+1)$ is composite.

**Solution to problem III-1-S.2** *Show that the arithmetic progression* 1, 5, 9, 13, 17, 21, 25, 29, 33, ...*contains infinitely many prime numbers.*

Another way of expressing this is: *Show that there are infinitely many primes of the form* $4k+1$. It so happens that the corresponding problem with $4k-1$ instead of $4k+1$ is easier to solve. This is because of the following property: *The product of numbers all of the form* $4k+1$ *is also of that form.*

From this it follows: *If an odd positive integer n is of the form* $4k-1$, *then it has at least one prime factor of that form*.

Now consider the primes of the form $4k-1$. They are: 3, 7, 11, 19, .... Suppose there is a last such prime, say $p$. Now construct the following number $n$:

$$n = 4(3 \times 7 \times 11 \times \cdots \times p) - 1.$$

This is of the form $4k-1$, so it has a prime factor $q$ of this form. The prime $q$ cannot be any of 3, 7, 11, ..., $p$, as $n$ is not divisible by these primes. So we have found a new prime of the form $4k-1$. Hence there cannot be a 'last prime' of this form. Therefore there are infinitely many primes of the form $4k-1$.

This method of proof does not work for primes of the form $4k+1$ because we *cannot* make a statement like this one: 'If $n$ is of the form $4k+1$, then it has at least one prime factor of that form.' (An easy counterexample to this hypothesis is the number $21 = 3 \times 7$.) Some other approach is needed. This 'other approach' is provided by the following at-first-sight-surprising fact which we do not prove here: *The prime factors of a number of the form* $4m^2+1$ *are all of the form* $4k+1$. Examples: $4 \times 4^2 + 1 = 65 = 5 \times 13$, and $4 \times 6^2 + 1 = 145 = 5 \times 29$. Taking this to be a fact, the rest of the proof is easy.

Suppose that there is a last prime $p$ of the form $4k+1$. We now construct the number $n = 4(5 \times 13 \times 17 \times \cdots \times p)^2 + 1$. The prime factors of $n$ are all of the form $4k+1$ and distinct from 5, 13, 17, ..., $p$. Hence $p$ cannot be the last such prime. So there are infinitely many such primes.

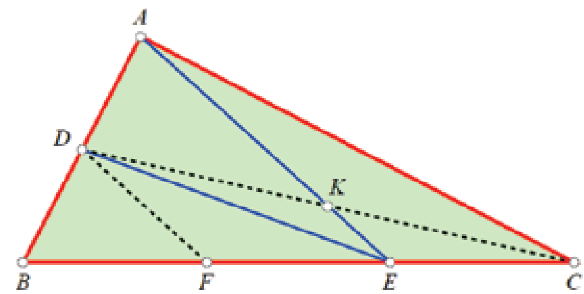


FIGURE 1.

**Solution to problem III-1-S.3** *In* $\triangle ABC$, *the midpoint of AB is D, and E is the point of trisection of BC closer to C. Given that* $\measuredangle ADC = \measuredangle BAE$, *determine the magnitude of* $\measuredangle BAC$.

Let $K$ be the point of intersection of $CD$ and $AE$. (See Figure 1.) Observe that in triangle $AKD$, $\measuredangle KAD = \measuredangle KDA$. Hence $DK = AK$. Let $F$ be the midpoint of $BE$. Note that $DF$ is parallel to $AE$. In triangle $CDF$, $E$ is the midpoint of $CF$, and $EK$ is parallel to $DF$. Therefore $K$ is the midpoint of $CD$. Hence in triangle $ACD$, $CK = DK = AK$. It follows that $\measuredangle CAD = \measuredangle BAC = 90°$.

**Solution to problem III-1-S.4** *Given a* $\triangle ABC$, *does there necessarily exist a point D on side BC such that* $\triangle ABD$ *and* $\triangle ACD$ *have equal perimeter? If such a point D exists, then we can similarly obtain points E and F on AC and AB, respectively, such that BE and CF bisect the perimeter of ABC. Are the lines* $AD, BE, CF$ *concurrent?*

Let $BC = a$, $CA = b$ and $AB = c$. Let $D$ be a point on $BC$, between $B$ and $C$, such that $BD = x$ and $CD = y$. $AD$ bisects the perimeter of triangle $ABC$ if and only if $x + c = y + b = s$, where $2s = a + b + c$. Thus $x = s - c$ and $y = s - b$. Since $s - c$ and $s - b$ are positive quantities whose sum is $a$, it is possible to find a point $D$ on $BC$ such that $BD = x$ and $CD = y$. See Figure 2. (More precisely, $D$ is the point where the ex-circle opposite vertex $A$ touches $BC$.)

For the second part, concurrency of the three line segments $AD$, $BE$, $CF$ follows from the converse of Ceva's theorem. (For we have, in the same way: $CE = s - a$, $EA = s - c$, $AF = s - b$, $FB = s - a$.

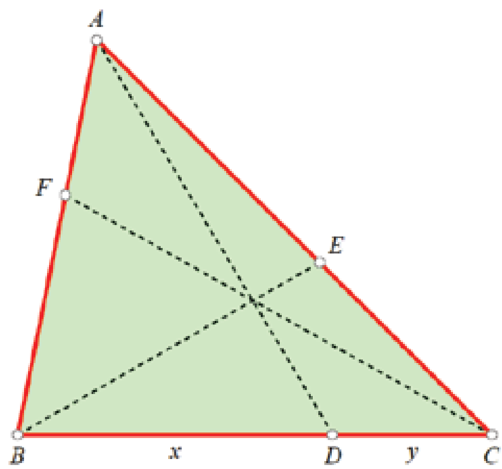


FIGURE 2.

- $BC = a, CA = b, AB = c$
- $c + x = b + y = s$
- $x = s - c$
- $y = s - b$

*EA = s – c, AF = s – b, FB = s – a.*

Hence

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1.$$

The converse of Ceva's theorem states that if $D, E, F$ are points on $BC, CA, AB$ such that the above equality holds, then $AD, BE, CF$ concur. Hence the claim.)

**Solution to problem III-1-S.5** *Let* $A = 5^{2013}$ *and* $B = 4^{2013}$. *Is* $4^A + 5^B$ *a prime number?*

Observe that 4 divides $A - 1$. Let $A - 1 = 4m$. Now

$$4^A + 5^B = 4(4^m)^4 + (5^{4^{2012}})^4$$

is of the form $4x^4 + y^4$. But:

$$4x^4 + y^4 = \left[(y - x)^2 + x^2\right]\left[(y + x)^2 + x^2\right].$$

Put $x = 4^m$ and $y = 5^{4^{2012}}$. Since both factors exceed 1, $4^A + 5^B$ is composite.