in the classroom

# How To Prove It

*This continues the 'Proof' column begun in the last issue. In this 'episode' too we study some problems from number theory; more specifically, from patterns generated by sums of consecutive numbers.*

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## Sums of consecutive numbers

Few of us would be impressed by the relation

$$1 + 2 = 3,$$

but when we set it alongside the following:

$$4 + 5 + 6 = 7 + 8,$$

our eyebrows go up a bit. And they climb up very much further if we list the following:

$$9 + 10 + 11 + 12 = 13 + 14 + 15.$$

At this point the mathematician in us will surely demand the clear statement of some general relation, and its proof as well. Let us respond to this challenge.

Note the sequence of first numbers in these relations: 1, 4, 9, .... It is clear that the first number in the $n^{th}$ relation is $n^2$. Noting the *number* of numbers on the left side and the right side in the relations (2, 3, 4, ...on the left side, and 1, 2, 3, ...on the right side), it appears that we are claiming the following:

> *For each positive integer $n$, the sum of $n + 1$ consecutive numbers starting with $n^2$ is equal to the sum of the next $n$ consecutive numbers.*

***Keywords:*** *Sequence, consecutive number, generalization, triangular number, pictorial*

For $n = 4$ the claim is that $16 + 17 + 18 + 19 + 20$ is equal to $21 + 22 + 23 + 24$, and this is true (each sum is 90). How do we check whether this claim is true for every $n$?

Let's look more closely at the statements. In the statement relating $9 + 10 + 11 + 12$ to $13 + 14 + 15$, note that on the left side the first number in the list is $9 = 3^2$ and the last number is $12 = 3^2 + 3$. In the statement relating $16 + 17 + 18 + 19 + 20$ to $21 + 22 + 23 + 24$, note that on the left side the first number in the list is $16 = 4^2$ and the last number is $20 = 4^2 + 4$. The pattern is clear: in the $n^{th}$ statement, on the left side the first number in the list is $n^2$, and the last number is $n^2 + n$. Also, there are $n + 1$ numbers. Using the well-known rule for the sum of the terms of an arithmetic progression ("half the sum of the first term and the last term, times the number of terms"), we see that the sum of the $(n + 1)$ numbers on the left side is

$$\frac{n^2 + (n^2 + n)}{2} \times (n + 1) = \frac{n(n + 1)(2n + 1)}{2}.$$

How about the sum on the right side? The first number in the list is clearly the number following the last number on the left side, and therefore it is $n^2 + n + 1$; and the last number is the one preceding the first number of the next such relation, i.e., the predecessor of $(n + 1)^2$; hence it is $(n + 1)^2 - 1 = n^2 + 2n$. As there are $n$ numbers, their sum is

$$\frac{(n^2 + n + 1) + (n^2 + 2n)}{2} \times n = \frac{n(2n^2 + 3n + 1)}{2} = \frac{n(n + 1)(2n + 1)}{2}.$$

We have obtained the same expression as earlier, so the two sums are equal. Hence proved.

## A more informal approach

Here is a more informal way of arguing. Consider the two sets $\{9, 10, 11, 12\}$ and $\{13, 14, 15\}$. If we divide the last number (12) of the first set into three equal parts of 4 each ($12 \div 3 = 4$), and add one part to each of the other numbers in the set, we get, from 9, 10, 11, the numbers $9 + 4 = 13$, $10 + 4 = 14$, $11 + 4 = 15$. So it will naturally be the case that $9 + 10 + 11 + 12$ is equal to $13 + 14 + 15$.

Similarly, if we take the two sets $\{16, 17, 18, 19, 20\}$ and $\{21, 22, 23, 24\}$, divide the last number in the first set into four equal parts of 5 each and add one part to each of the other numbers in the set, we get, from $\{16, 17, 18, 19\}$ the set $\{21, 22, 23, 24\}$. So it will naturally be the case that $16 + 17 + 18 + 19 + 20$ is equal to $21 + 22 + 23 + 24$.

In the general case we have the two sets

$$A = \{n^2,\ n^2 + 1,\ n^2 + 2,\ \dots,\ n^2 + n\},$$
$$B = \{n^2 + n + 1,\ n^2 + n + 2,\ \dots,\ n^2 + 2n\}.$$

We take away the largest number $(n^2 + n)$ from $A$, divide it into $n$ parts of $n + 1$ each, and add this amount $(n + 1)$ to each of the remaining numbers; we get the set $A'$ given by:

$$A' = \{n^2 + n + 1,\ n^2 + n + 2,\ \dots,\ n^2 + 2n\},$$

which is exactly the set $B$. It follows that the sum of the numbers in $A$ is the same as the sum of the numbers in $B$.

## Triangular number identities

Triangular numbers offer a rich environment for exploration of number patterns and identities. Bring a group of youngsters to this fertile ground, and you will soon have a few discoveries on your hands, including some you may not have seen earlier.

The triangular numbers ("T-numbers" for short) are the numbers $1, 1 + 2 = 3, 1 + 2 + 3 = 6, 1 + 2 + 3 + 4 = 10, \dots$; thus, they are the partial sums of the sequence of natural numbers. The $n^{th}$ such number is denoted by $T_n$ $(n = 1, 2, 3, \dots)$:

$$T_n = 1 + 2 + 3 + 4 + \cdots + (n - 1) + n,$$

or in summation notation: $T_n = \sum_{k=1}^{n} k$. Here are the first ten T-numbers:

$T_1 = 1,\quad T_2 = 3,\quad T_3 = 6,\quad T_4 = 10,$
$T_5 = 15,\quad T_6 = 21,\quad T_7 = 28,$
$T_8 = 36,\quad T_9 = 45,\quad T_{10} = 55.$

It is well known that

$$T_n = \frac{n(n + 1)}{2},$$

and there are several ways of proving this relation alone. Here are some pretty relations that children quickly discover for themselves:

**1** *The sum of any two consecutive T-numbers is a perfect square.*

For example, $T_1 + T_2 = 4 = 2^2$ and $T_6 + T_7 = 49 = 7^2$.

**2** *If* $x$ *is a T-number, then* $8x + 1$ *is a perfect square. Conversely, if* $8x + 1$ *is an odd perfect square, then* $x$ *is a T-number. Otherwise expressed: if* $x$ *is an odd perfect square, then* $(x - 1)/8$ *is a T-number.*

For example, 3 is a T-number, and $8 \times 3 + 1 = 25 = 5^2$ is a perfect square. Similarly, 81 is an odd perfect square, and $(81 - 1)/8 = 10$ is a T-number.

**3** *If* $x$ *is a T-number, then so is* $9x + 1$.

For example, take the T-numbers 3, 10 and 36. Multiplying them by 9 and adding 1 we get the numbers 28, 91 and 325. It remains to check that each of these is a T-number; indeed, they are: $28 = T_7$, $91 = T_{13}$ and $325 = T_{25}$.

The next challenge is to get the children to find *proofs* of these various relations. We now take up this theme.

## "The sum of two consecutive T-numbers is a perfect square.":

Many different approaches are possible. Perhaps the most direct way is the one based on 'pure algebra'. We know that $T_n = \frac{1}{2}n(n+1)$. So:

$$T_{n-1}+T_n = \frac{1}{2}(n-1)n+\frac{1}{2}n(n+1) = \frac{1}{2}n(n-1+n+1)$$

$$= \frac{1}{2}n \times 2n = n^2.$$

Hence the sum of the $(n-1)^{th}$ and $n^{th}$ triangular numbers is the $n^{th}$ perfect square.

*Other approaches:* But it is fun to seek other ways. Here is a way which draws on the definition of $T_n$ as the sum of the first $n$ positive integers together with the well-known and often-used fact that the sum of the first $n$ odd positive integers equals $n^2$. These two facts acting in concert with another simple fact —that each odd number is the sum of two consecutive numbers (e.g., $5 = 2 + 3$) —yield a nice proof; all we need to do is re-bracket the numbers and add them in a slightly different order. We illustrate the idea for $n = 3$:

$$\begin{aligned} 3^2 &= 1+3+5 = (0+1)+(1+2)+(2+3) \\ &= (0+1+2)+(1+2+3) \\ &= T_2 + T_3. \end{aligned}$$

Similarly, consider $n = 5$:

$$\begin{aligned} 5^2 &= 1+3+5+7+9 \\ &= (0+1)+(1+2)+(2+3)+(3+4)+(4+5) \\ &= (0+1+2+3+4)+(1+2+3+4+5) \\ &= T_4 + T_5. \end{aligned}$$

Without having to elaborate on the details, it should be clear that such a re-arrangement of summands will always work. But for those who are keen on seeing how the idea can be expressed symbolically, here is how we do it:

$$\begin{aligned} n^2 &= \sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n}\big((k-1)+k\big) \\ &= \sum_{k=1}^{n}(k-1) + \sum_{k=1}^{n} k \\ &= \sum_{k=1}^{n-1} k + \sum_{k=1}^{n} k \\ &= T_{n-1} + T_n. \end{aligned}$$

*A pictorial way:* There's even a way of expressing the relation using pictures! We regard the numbers $T_{n-1}$ and $T_n$ as representing the areas of two staircase-shaped polygons as depicted in Figure 1, which show the polygons for $n = 6$. Observe how neatly they fit together to form a $6 \times 6$ square.

Though the construction has been shown only for the specific case $n = 6$, it is not hard to see that the same idea will work for any $n$.

## "8 times a T-number plus 1 is a perfect square."

Here is an algebraic proof. We know that $T_n = \frac{1}{2}n(n+1)$. So if $x$ is a T-number then $x = \frac{1}{2}n(n+1)$ for some positive integer $n$.

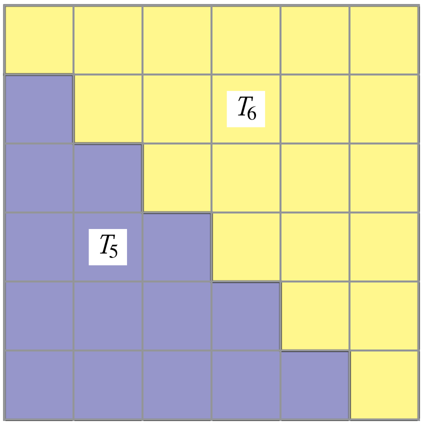


Figure 1. Illustrating "why" $T_5 + T_6 = 6^2$

But in this case we have:

$$8x + 1 = 8 \times \frac{n(n+1)}{2} + 1 \;=\; 4n(n+1) + 1$$
$$= 4n^2 + 4n + 1$$
$$= (2n+1)^2.$$

*Proof of the converse:* Suppose that $8x + 1$ is an odd perfect square. Then $8x + 1 = (2n+1)^2$ for some integer $n$. Hence:

$$x = \frac{(2n+1)^2 - 1}{8} = \frac{4n^2 + 4n}{8} = \frac{n(n+1)}{2} = T_n.$$

This property provides a simple way of checking whether a given number $x$ is a T-number. (Example: The number 3003 is a T-number, because $8 \times 3003 + 1 = 24025 = 155^2$.)

*A pictorial way:* As earlier there is an elegant way of depicting the relation "If $x$ is a T-number, then $8x + 1$ is a perfect square". We know that if $x$ is a T-number, then $x = \frac{1}{2}n(n+1)$ for some integer $n \geq 0$. Otherwise put, if $x$ is a T-number, then $2x = n(n+1)$ for some integer $n \geq 0$. Hence we may associate with $2x$ a rectangle of dimensions $n \times (n+1)$. Note that the length of this rectangle exceeds the breadth by 1 unit. (This fact has a bearing on the outcome as we shall see shortly.) Four such rectangles may be fitted together as shown in Figure 2, in which we have taken $n = 3$.

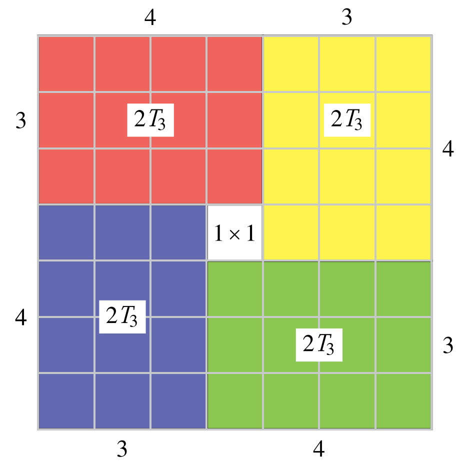


Figure 2. Illustrating 'why' $8T_3 + 1$ is a perfect square

The four rectangles neatly enclose a square measuring $1 \times 1$ in the centre, since $4 - 3 = 1$. We see from this that $8T_3 + 1$ is a perfect square, indeed, $8T_3 + 1 = (4+3)^2$. In general we have: $8T_n + 1 = \left((n+1) + n\right)^2$.

## The third property

It remains to show this: *If $x$ is a T-number, then so is $9x + 1$.* But we shall leave the proof to you. (*Hint.* Use the property: "$n$ is a T-number if and only if $8n + 1$ is a perfect square".) Indeed we shall challenge you with the following problem:

> *The pair of integers $a = 9$ and $b = 1$ has the property that if $x$ is a T-number, then so is $ax + b$. Find all such pairs of integers.*

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