in the classroom

How to …

Solve a Geometry Problem

Part–2

Continuing our informal, short self-help guide on solving geometry problems. In the second part of this series, Ajit Athle describes some strategies which help in solving geometry problems and demonstrates how these strategies are used in solving an intriguing problem.

Ajit Athle

George Pólya once remarked, "Geometry is the science of correct reasoning on incorrect figures." But drawing an accurate figure is often an important first step in solving a problem in geometry, because it may reveal an unsuspected relationship — perhaps an equality of a pair of angles, or a pair of sides, or the perpendicularity of a pair of sides; the possibilities are many. In the same vein the use of colour can help — marking different parts of the figure in different ways. Any approach is permissible if it helps you to spot relationships which are otherwise nearly invisible. In this edition of 'Geometry Corner' we solve a challenging and intriguing problem.

An incircle & median problem

In $\triangle ABC$, the median AM to side BC is trisected by the incircle, i.e., $AP = PQ = QM$. Find the ratios $AB : BC : CA$. (See Figure 1.) Try to solve the problem before reading on.

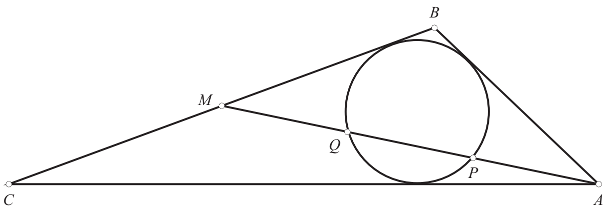

FIGURE 1. Problem concerning an incircle and a median

Solution to the problem

Let E, F, G be the points of contact of the incircle with the sides AB, BC, CA of the triangle (Figure 2).

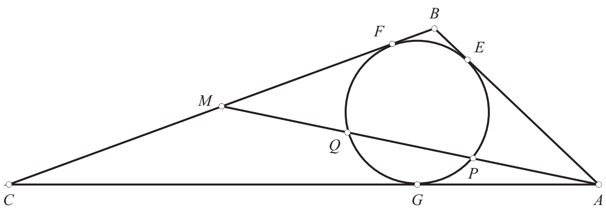

FIGURE 2.

A good beginning would be to take $BC = 2a$, as that would make $BM = MC = a$. Let us also take $BF = d$. By the equal tangents theorem, $BE = d$. By the Power-point theorem,

$$MF^2 = MQ \cdot MP, \qquad AE^2 = AP \cdot AQ,$$

and since $AP = MQ$ and $MP = AQ$, we have $MF = AE$. Let $AP = m$; then $MQ = QP = m$, hence $MF^2 = 2m^2$, and $MF = m\sqrt{2}$. But $MF = a - d$ as well, therefore:

$$(a-d)^2 = 2m^2. \qquad (1)$$

Further, as AM is a median we may employ the theorem of Apollonius to $\triangle ABC$ to give:

$$\begin{aligned} AB^2 + AC^2 &= 2\left(BM^2 + AM^2\right) \\ &= 2\left(a^2 + 9m^2\right). \qquad (2) \end{aligned}$$

Since $AE = MF = a - d$, it follows that $AB = a$. Also,

$$\begin{aligned} AC &= AG + CG = AE + CF \\ &= a - d + 2a - d = 3a - 2d. \end{aligned}$$

Substituting for AB and AC in (2) and then combining (1) and (2), we obtain the following quadratic equation:

$$5d^2 - 6ad + a^2 = 0.$$

This is easily factorized and solved to yield:

$$d = a, \quad \text{or} \quad d = \frac{a}{5}.$$

Of these, the former has to be rejected as it is inconsistent with the given conditions (it would make $MF = 0$). Hence, $d = a/5$ and this yields $AC = 3a - 2a/5 = 13a/5$. Therefore:

$$AB = a, \quad BC = 2a, \quad CA = \frac{13a}{5},$$

giving us the required proportions, $AB : BC : CA = 5 : 10 : 13$.

Note how knowledge of the equal tangents theorem, the intersecting chords theorem (or “power of a point” theorem) and Apollonius’ theorem helped us to arrive at the answer.

Appendix: Some standard theorems of plane geometry

Equal tangents theorem: *Given a circle $\mathcal{C}$ and a point P outside the circle, let PA and PB be the two tangents that can be drawn from P to $\mathcal{C}$. Then $PA = PB$.*

The theorem has a natural extension to three dimensions, with 'sphere' taking the place of 'circle'.

Intersecting chords theorem: *Given a circle $\mathcal{C}$, let two chords AB and CD meet at a point P. Then $PA \cdot PB = PC \cdot PD$.*

Remark. The result is true even if P lies outside the circle, or if one of the chords is tangent to the circle. The theorem has a natural converse. The value of $PA \cdot PB$ is called the *power of P with respect to the circle $\mathcal{C}$*.

Theorem of Apollonius: *Given a triangle ABC, let D be the midpoint of BC. Then $AB^2 + AC^2 = 2\left(AD^2 + BD^2\right)$.*

The result follows easily from the Pythagorean theorem; see if you can prove it. There are also easy and natural proofs using vectors; using coordinate geometry; and using trigonometry. The theorem has a generalization called Stewart's theorem.

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