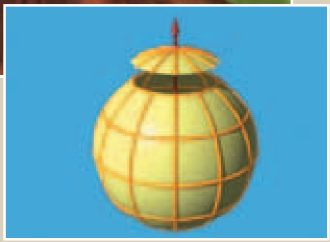


review

Review of ‘Dimensions’

Dheeraj Kulkarni

The animation movie series “Dimensions” produced by Jos Leys, Étienne Ghys and Aurélien Alvarez is a free-to-download series available at the website http://www.dimensions-math.org. It is a highly refined exposition on the notion of *dimension* and the ideas centered round it. A beautiful combination of step-by-step build-up of mathematical ideas, highly creative use of animation and graphics, accompanied by melodious background music, makes it a unique exposition, perhaps one of the best expositions of mathematics available, and accessible to a wide audience. An appealing device that the authors use is the dramatized narration by actual mathematicians (present and past).

The series consists of nine chapters. Each chapter is just fourteen minutes long and builds on the preceding chapters. The series is densely packed with mathematical ideas and so provides a learning opportunity for students and teachers as well. Teachers can take away a wealth of experience on how to convey mathematical ideas.

We briefly describe the contents of each chapter. Chapter 1 starts with dimension two or 2D. In this chapter, Hipparchus explains that a pair of numbers is enough to describe the position of a point on the sphere. To describe a point on the earth, two parameters, namely the longitude and the latitude, are needed.

This is demonstrated with the help of animation of a tiny plane travelling on the surface of the earth. To draw an earth map using a globe on a table, we draw a ray from the North Pole (NP) to a point on the globe and locate its point of intersection with the plane of the table. This is known as the 'stereographic projection' of the point *p* onto the plane, and the process is known as 'stereographically projecting the sphere onto the plane'. See Figures 1 and 2.

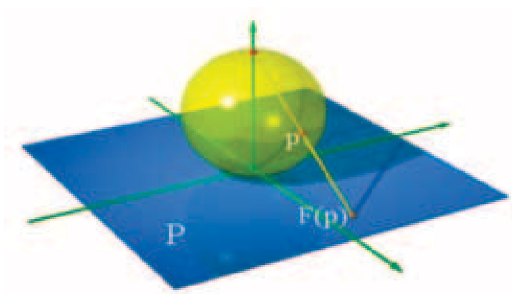


Figure 1. Stereographic projection

The film shows a wonderful animation of stereographic projections. One can see that the North Pole has no stereographic projection on the surface of the table, thus we say that it lies 'at infinity'. Note that stereographic projection does not preserve size. But angles between intersecting lines are preserved, hence directions are preserved. This gives an intuitive idea of the property of 'conformality' of this projection, i.e., 'preserving angles between curves'. The stereographic projection takes meridians to radii emanating from South Pole, and parallels to the concentric circles centered at the South Pole.

Figure 2. Stereographic projection of Earth

In Chapter 2, M.C. Escher tells us how 2D creatures on flat 2D surfaces might imagine 3D objects, giving us a hint on how to imagine 4D objects. The idea is to understand the three dimensions using cross sections on a surface, as though the 3D surface is passing through the flat 2D surface of the table (see Figure 3). To develop an understanding of platonic solids only through cross sections, we use the stereographic projections. By counting the number of the faces and edges and vertices, the creatures on the flat surface can develop an understanding of the 3D platonic solids. This is well demonstrated in the movie. This method of using cross sections helps prepare our imagination to understand the fourth dimension by using 3D cross sections of 4D objects.

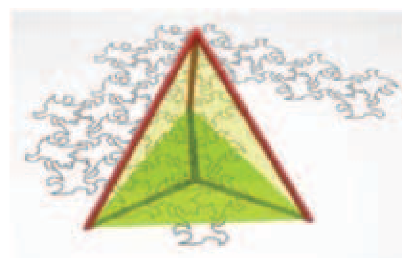


Figure 3. Cross section of a 3D Platonic solid

In Chapters 3 and 4, Swiss mathematician Ludwig Schläfli talks to us about imagining objects in 4D. He was one of the first to study geometry in higher dimensions. He shows us the idea of representing points in higher dimensional space by tuples of real numbers using a magic blackboard. Generalizing the idea of a line segment (a 1D simplex), an equilateral triangle (a 2D simplex) and a regular tetrahedron (a 3D simplex), he obtains a 4D 'simplex' on the board. To get a feel of a 4D object we must see 3D cross sections of the object like the flat creatures did in Chapter 2. The key idea is to describe the number of vertices, edges and 2D and 3D faces to describe the 4D simplex uniquely. In Chapter 4 we see stereographic projections of 4D simplexes to 3D space. We see faces blowing up because of rotation of the simplexes in 4D!

In Chapters 5 and 6, French mathematician Adrien Douady explains the notion of complex numbers and geometrical transformation of the plane in an interesting way.

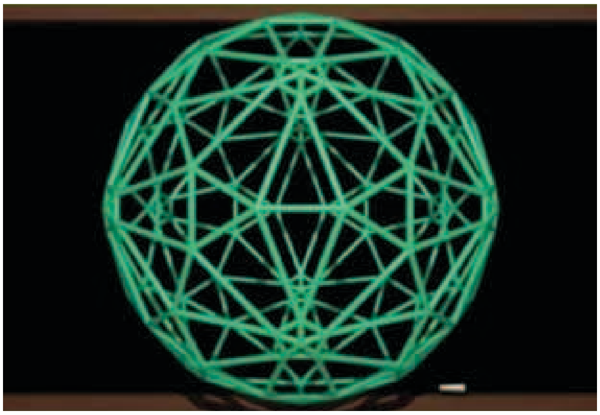


Figure 4. 3D cross section of a "600" simplex in 4D

The square root of –1 may be regarded as ¼ of a full turn since multiplication by –1 may be regarded as a half-turn. Once we understand this idea geometrically, we can visualize the addition and multiplication of complex numbers. This is shown with specific examples.

Combining these two notions – stereographic projection and complex numbers – it is seen that complex numbers are sufficient to describe all points on the sphere except the North Pole.

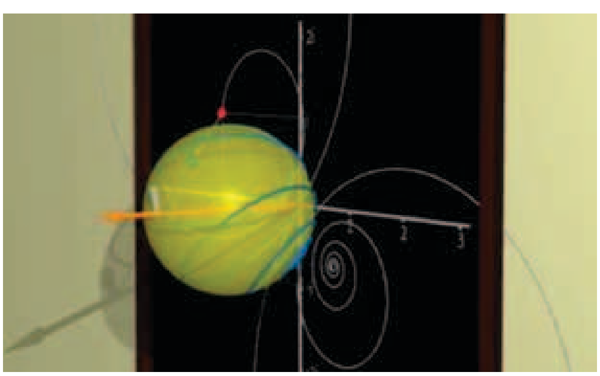
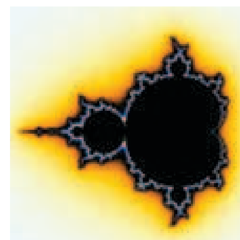


Figure 5. Sphere as a 'Complex Projective Line'

In Chapter 6, Douady describes transformations of the plane – dilations and rotations – using complex numbers. We see Douady's photograph getting dilated and rotated under the transformations. Similarity transformations are neatly encoded by complex numbers. We observe that even after the transformation we recognize Douady! Small shapes such as eyes and buttons preserve their shape. This is the geometric idea behind conformal maps.

The chapter then moves to studying dynamics of iterations, that is, the effects of applying the same transformation repeatedly; we see what happens to sets in the complex plane. This is nicely shown through animation. The famous Mandelbrot set is shown, and fractal structures discussed. One marvels at the beauty of the sets. Through iteration of simple functions one can produce rich and intricate structures.



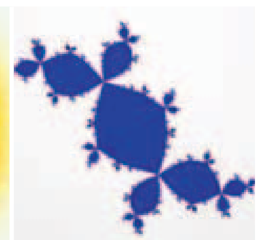


Figure 6. Mandelbrot Set Figure 7. A Julia Set

In Chapters 7 and 8, mathematician Heinz Hopf describes the strange and non-intuitive idea of a 'fibration' using complex numbers. He shows us a beautiful arrangement of circles which together form a 3D sphere.

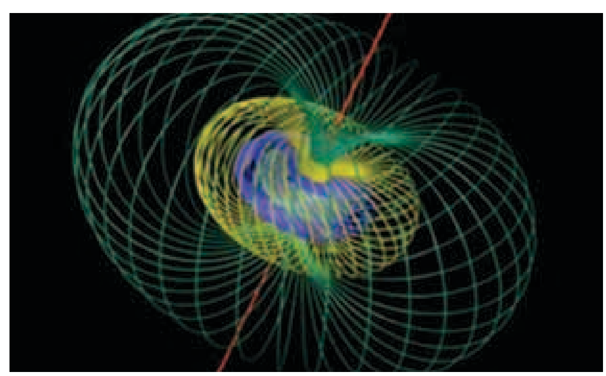


Figure 8. Hopf fibration

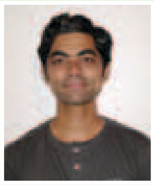
Hopf discovered one of the most important and well understood fibrations, known today as the 'Hopf fibration'. It is a smooth arrangement of circles, no two of which cut one another, forming a 3D sphere; each circle corresponds to a point in the 2D sphere.

In Chapter 8 we get a closer look at the Hopf fibration. The idea of formal proof is presented in Chapter 9 by mathematician Bernhard Riemann.

The chapter details can be found at http://www.dimensions-math.org/Dim_chap_E.htm. All images in this review are from the site http://www.dimensions-math.org/Dim_E.htm.

The author has the following suggestions for teachers. Since the series is densely packed with ideas, the series can be screened over the period of nine weeks, each week devoted to a chapter; the screening may be followed by a discussion of the main ideas. It may be helpful to pause and replay the video to understand the subtlety of ideas and their highly accurate presentations by the creators. Teachers will need to study the background material (found at the link given above) for the chapter before the screening.

For both students and teachers alike, the series will be an enjoyable journey to explore the notion of dimension. It will give a glimpse into how mathematical ideas are developed.



Dheeraj Kulkarni completed his Ph.D. in Mathematics from the Indian Institute Of Science, Bangalore, in 2012. He is currently visiting Georgia Tech, Atlanta, USA as a Post Doctoral Fellow. Apart from academics he enjoys sports such as football, cricket, swimming and long distance cycling. He also enjoys listening to music and going for long walks. He may be contacted at dheeraj.kulkarni@gmail.com.