Problems for the Senior School

Problem editors: PRITHWIJIT DE & SHAILESH SHIRALI

problem corner

Problems for Solution

Problem II-2-S.1
A circle has two parallel chords of length x that are x units apart. If the part of the circle included between the chords has area $2 + \pi$, find the value of x.

Problem II-2-S.2
The prime numbers p and q are such that $p + q$ and $p + 7q$ are both perfect squares. Determine the value of p.

Problem II-2-S.3
Determine the value of the infinite series

$$\frac{1}{3^2+1} + \frac{1}{4^2+2} + \frac{1}{5^2+3} + \frac{1}{6^2+4} + \cdots.$$

Problem II-2-S.4
In trapezium $ABCD$, the sides AD and BC are parallel to each other; $AB = 6$, $BC = 7$, $CD = 8$, $AD = 17$. Sides AB and CD are extended to meet at E. Determine the magnitude of $\measuredangle AED$.

Problem II-2-S.5
You are told that the number 27000001 has exactly four prime factors. Find their sum. (Computer solution not acceptable!)

Solutions of Problems in Issue-II-1

Solution to problem I-2-S.1 *Drawn through the point A of a common chord AB of two circles is a straight line intersecting the first circle at the point C, and the second circle at the point D. The tangent to the first circle at the point C and the tangent to the second circle at the point D intersect at the point M. Prove that the points M, C, B, and D are concyclic.* (See Figure 1.)

Two cases are possible: (i) C and D are on the same side of the line joining the centres of the circle. (ii) C and D are on the opposite sides of the line joining the centres of the circle. In both cases we see that $\measuredangle MCD = \measuredangle CBA$ and $\measuredangle MDC = \measuredangle ABD$. Thus $\measuredangle CBD = \measuredangle CBA + \measuredangle ABD = \measuredangle MCD + \measuredangle MDC = 180^\circ - \measuredangle CMD$. Therefore points M, C, B, and D are concyclic.

Solution to problem I-2-S.2 *In triangle ABC, point E is the midpoint of the side AB, and point D is the foot of the altitude CD. Prove that $\measuredangle A = 2\measuredangle B$ if and only if $AC = 2ED$.* (See Figure 2.)

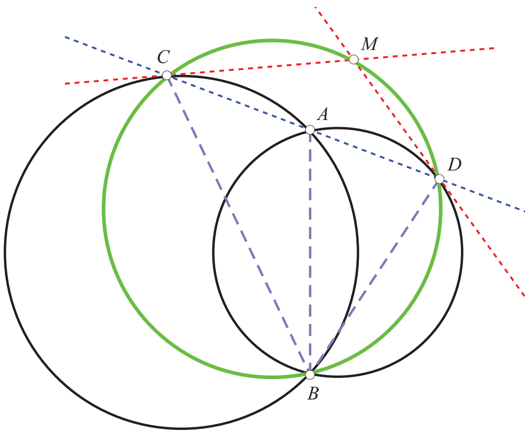

Figure 1. Showing that M, C, B, D are concyclic (Problem I-2-S.1)

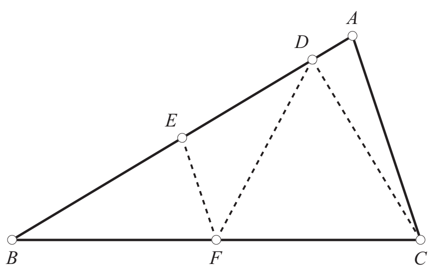

Figure 2. Showing that $\measuredangle A = 2\measuredangle B$ if and only if $AC = 2ED$ (Problem I-2-S.2)

Let F be the midpoint of BC. Therefore $EF \parallel AC$, and $AC = 2EF$. Also in right-angled $\triangle CDB$, F is the midpoint of the hypotenuse BC. Therefore $CF = DF = BF$.

In $\triangle BFD$, $DF = BF$. So $\measuredangle FDB = \measuredangle FBD = \measuredangle B$. Since $EF \parallel AC$, $\measuredangle FEB = \measuredangle A$. But $\measuredangle FEB = \measuredangle FDB + \measuredangle DFE$. That is, $\measuredangle A = \measuredangle B + \measuredangle DFE$. Now

$$\measuredangle A = 2\measuredangle B \iff \measuredangle DFE = \measuredangle B \iff EF = ED \iff AC = 2EF = 2ED.$$

Solution to problem I-2-S.3 *Solve the simultaneous equations: $ab + c + d = 3$, $bc + d + a = 5$, $cd + a + b = 2$, $da + b + c = 6$, where a, b, c, d are real numbers.*

Adding the four equations we obtain

$$(a + c)(b + d) + 2(a + c) + 2(b + d) = 16. \quad (1)$$

Adding the first two equations we obtain

$$(b + 1)(a + c) + 2d = 8. \quad (2)$$

Adding the last two equations gives

$$(d + 1)(a + c) + 2b = 8. \quad (3)$$

Subtracting (3) from (2) yields $(a + c - 2)(b - d) = 0$. Thus either $a + c = 2$ or $b = d$. But, if $b = d$ then $bc + d + a = cd + a + b$ which leads to $5 = 2$, an absurdity. Therefore $a + c = 2$. Now from (1) we get $b + d = 3$. So $c + d = 5 - (a + b)$. Therefore:

$$3 = ab + c + d = ab + 5 - (a + b), \quad (4)$$
$$2 = cd + a + b = (2 - a)(3 - b) + a + b$$
$$= 6 + ab - 2a - b. \quad (5)$$

These lead to:

$$a + b - ab = 2, \quad (6)$$
$$2a + b - ab = 4. \quad (7)$$

From (6) and (7), $a = 2$. Hence $c = 2 - a = 0$. Therefore $b = 2 - a - cd = 0$ and $d = 3 - b = 3$. It is easy to see that these values satisfy the given equations. Therefore $(a, b, c, d) = (2, 0, 0, 3)$.

Solution to problem I-2-S.4 *Let x, y, a be positive numbers such that $x^2 + y^2 = a$. Determine the minimum possible value of $x^6 + y^6$ in terms of a.*

We have: $x^6 + y^6 = (x^2 + y^2)\{(x^2 + y^2)^2 - 3x^2y^2\} = a(a^2 - 3x^2y^2)$. Hence $x^6 + y^6$ attains its minimum value when x^2y^2 attains its maximum value. Since $x^2 + y^2 = a$, the maximum possible value of x^2y^2 is $(a/2)^2 = a^2/4$, with equality attained only when $x = y = \sqrt{a/2}$. Hence the minimum value of $x^6 + y^6$ is $a\left(a^2 - 3a^2/4\right) = a^3/4$.

Solution to problem I-2-S.5 *Let p, q and y be positive integers such that p is greater than q, and $y^2 - qy + p - 1 = 0$. Prove that $p^2 - q^2$ is not a prime number.*

Suppose $p^2 - q^2 = (p + q)(p - q)$ is a prime number. Then $p - q = 1$ and therefore $y^2 - qy + q = 0$. So $q = y^2/(y - 1) = y + 1 + 1/(y - 1)$.

As q and y are integers, so is $1/(y - 1)$. Since $y \geq 1$, it must be that $y = 2$. Hence $q = 4$ and $p = q + 1 = 5$, giving $p^2 - q^2 = 9$, which is not prime. A contradiction. Therefore $p^2 - q^2$ is not a prime number.