A Relation between Polynomial and Exponential Functions

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In this note, we hit upon an interesting property of polynomial functions that mimic the behavior of exponential functions.

Statement of the problem. Consider the following series of questions about a polynomial $P(x)$ which mimics the behaviour of an exponential function.

(1) Suppose that P is quadratic, and $P(x) = 2^x$ for $x = 0, 1, 2$. What is the value of $P(3)$?

(2) Suppose that P is cubic, and $P(x) = 2^x$ for $x = 0, 1, 2, 3$. What is the value of $P(4)$?

(3) Suppose that P is quadratic, and $P(x) = 3^x$ for $x = 0, 1, 2$. What is the value of $P(3)$?

(4) Suppose that P is cubic, and $P(x) = 3^x$ for $x = 0, 1, 2, 3$. What is the value of $P(4)$?

(5) Suppose that P is a polynomial in x of degree n, and $P(x) = 2^x$ for $x = 0, 1, 2, \ldots, n$. What is the value of $P(n+1)$?

(6) Suppose that P is a polynomial in x of degree n, and $P(x) = 3^x$ for $x = 0, 1, 2, \ldots, n$. What is the value of $P(n+1)$?

We shall prove an elegant result here which answers all such questions at once.

Theorem. Let $P(x)$ be a polynomial in x of degree n. Suppose that $P(x) = a^x$ for $x = 0, 1, 2, \ldots, n$, for some number a. Then $P(n+1) = a^{n+1} - (a-1)^{n+1}$.

So if P is quadratic, and $P(x) = 3^x$ for $x = 0, 1, 2$, then (going by the theorem) the value of $P(3)$ is $3^3 - 2^3 = 19$.

Keywords: Polynomial function, exponential function

Let us check this claim the 'long way' – by actually computing the expression for $P(x)$. Let $P(x) = a + bx + cx^2$. Then we have:

$$\begin{aligned} a &= 3^0 = 1, \\ a + b + c &= 3^1 = 3, \\ a + 2b + 4c &= 3^2 = 9. \end{aligned}$$

The first two equations yield $b + c = 2$, and the first and third equations yield $2b + 4c = 8$. From these we get $c = 2$ and $b = 0$. Hence $P(x) = 1 + 2x^2$, so $P(3) = 19$, as claimed.

Proof. We shall prove the claim using the principle of induction, by induction on the degree of the polynomial.

Let us first establish the claim for polynomials of degree 1 (i.e., linear polynomials). Let $P(x)$ be a polynomial in x of degree 1, and suppose that $P(0) = 1$, $P(1) = a$. Then the claim is that $P(2) = a^2 - (a-1)^2$, i.e., $P(2) = 2a - 1$. To prove this, note that since $P(x)$ is of degree 1, we have

$$P(2) - P(1) = P(1) - P(0), \qquad \therefore\ P(2) = 2P(1) - P(0) = 2a - 1,$$

as claimed.

The key result used is the following.

Lemma. *Let $f(x)$ be a polynomial in x of degree n, where n is a positive integer. Define $g(x)$ by:*

$$g(x) = f(x+1) - f(x). \tag{1}$$

Then $g(x)$ is a polynomial in x of degree $n - 1$.

For example, if $f(x) = x^3$, a polynomial of degree 3, then $g(x) = (x+1)^3 - x^3 = 3x^2 + 3x + 1$, a polynomial of degree 2. (Editor's note. It should be clear why this is true: the highest degree term in $f(x)$ gets cancelled as a result of the subtraction, so the degree of g is lower than that of f. In fact, the degree of g is $n - 1$, which is 1 lower than the degree of f. We shall say more about this lemma in the appendix.)

The induction hypothesis is the following.

> Suppose that $f(x)$ is a polynomial in x of degree k (a positive integer), such that $f(x) = a^x$ for $x = 0, 1, 2, \ldots, k$, for some a. Then $f(k+1) = a^{k+1} - (a-1)^{k+1}$.

To prove the induction step, we must assume the above and prove the following.

> Suppose that $g(x)$ is a polynomial in x of degree $k + 1$, such that $g(x) = a^x$ for $x = 0, 1, 2, \ldots, k, k+1$, for some a. Then $g(k+2) = a^{k+2} - (a-1)^{k+2}$.

Proof of the induction step. Let $g(x)$ be a polynomial in x with the stated properties: it has degree $k + 1$, and $g(x) = a^x$ for $x = 0, 1, 2, \ldots, k, k+1$, for some a. We must compute the value of $g(k+2)$. Now consider the function $h(x) = g(x+1) - g(x)$. We have:

$$\begin{aligned} h(0) &= g(1) - g(0) = a - 1, \\ h(1) &= g(2) - g(1) = a^2 - a = a(a-1), \\ h(2) &= g(3) - g(2) = a^3 - a^2 = a^2(a-1), \\ h(3) &= g(4) - g(3) = a^4 - a^3 = a^3(a-1), \end{aligned}$$

and in general:

$$h(x) = a^x(a-1) \qquad \text{for } x = 0, 1, 2, \ldots, k. \tag{2}$$

Define

$$f(x) = \frac{h(x)}{a-1}. \tag{3}$$

Then the function f satisfies the conditions of the induction hypothesis: it is a polynomial in x of degree k (by the lemma), and $f(x) = a^x$ for $x = 0, 1, 2, \ldots, k$. Hence $f(k+1) = a^{k+1} - (a-1)^{k+1}$.

Therefore we have, using the definitions of h and g:

$$\begin{aligned} h(k+1) &= (a-1) \cdot f(k+1) \\ &= (a-1) \cdot \left(a^{k+1} - (a-1)^{k+1}\right), \\ g(k+2) &= g(k+1) + h(k+1) \\ &= a^{k+1} + (a-1) \cdot \left(a^{k+1} - (a-1)^{k+1}\right) \\ &= a^{k+2} - (a-1)^{k+2}, \end{aligned}$$

as required. This proves the induction hypothesis. Hence the theorem is proved. □

Proof of the lemma

We must prove that if $f(x)$ is a polynomial of degree n, where n is a positive integer, and $g(x)$ is defined by $g(x) = f(x+1) - f(x)$, then $g(x)$ is a polynomial of degree $n-1$.

Proof

Let

$$f(x) = ax^n + bx^{n-1} + \cdots,$$

where $a \neq 0$. Then:

$$\begin{aligned} g(x) &= f(x+1) - f(x) \\ &= \left[a(x+1)^n + b(x+1)^{n-1} + \cdots\right] - \left[ax^n + bx^{n-1} + \cdots\right] \\ &= \left[a(x+1)^n - ax^n\right] + \left[b(x+1)^{n-1} - bx^{n-1}\right] + \cdots. \end{aligned}$$

In the expression $b(x+1)^{n-1} - bx^{n-1}$, the terms involving x^{n-1} cancel out, so the degree of that portion is less than $n-1$.

In the expression $a(x+1)^n - ax^n$, the terms involving x^n cancel out. The term involving x^{n-1} is ax^{n-1}, and this term survives, since $a \neq 0$ by assumption.

Hence the degree of g is $n-1$. □

Box 1

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