The path to mastery

Teacher: Leader, Supporter, Enabler

... is never smooth

Sneha Titus

It is often the kind teacher who is supportive and helpful, and ensures compliance using charm and persuasiveness, about whom effusive essays are written and who is remembered for the way he or she guided the students towards self-confidence and often, examination success. But how often does this teacher steer the student to mastery of the subject?

I was reminded of the stereotype of the kind teacher when I read an article by Alan Wigley[i] in which he describes two models for teaching mathematics. The first, *the path smoothing model*, is one practised by teachers who use *'the essential methodology of smoothing the path for the learner'*. This is how Alan Wigley describes this model:

1. *The teacher states the kind of problem on which the class will be working.*
2. *The teacher classifies the subject matter into a limited number of categories and presents them one at a time.*
3. *Pupils are led through a method for tackling the problems. The key principle is to establish secure pathways for the pupils. Thus it is important to present ways of solving problems in a series of short steps; often only one approach is considered seriously. Teachers question pupils, but usually in order to lead them in a particular direction.*
4. *Pupils work on exercises to practise the methods given aimed at involving learners more actively. These are usually classified by the teacher and graded for difficulty. Pupils repeat the processes until they can do so with the minimum of error.*
5. *Revision: Longer term failure is dealt with by returning to the same or similar subject matter throughout the course.*

Wigley goes on to explain that most teachers do provide insights into the concept they are teaching but under pressure to 'cover' the syllabus, they move on to the serious business of doing the exercises given. On reading this article, I was reminded of my experiences of teaching the chapter 'Maxima and Minima' in grades 11-12. I usually introduced the chapter with an interesting problem, such as the swimmer in distress who had to be reached in the least possible time by the life-guard. I would outline the stages of the solution and explain the theory at each stage. This would be followed by a series of problems solved in class, of increasing complexity. Having taught the chapter for many years, I was aware that there were some students who found this kind of problem rather difficult, though they were good at other sections of the course and had no difficulty with the topic of differentiation. Where they stumbled was in *understanding* what they were doing in the problem. Typically, the wording of the problem caused the difficulty: students could not distinguish between what they were given and what they had to prove. Once the problem was unfolded, they sped along the path to the solution.

I now see that the strategy I developed was a path smoothing model. Having recognized the boulder in the path (not the 'calculus' or 'small stone'!), I devised a series of steps which worked infallibly for all maxima and minima problems. I will use a familiar problem to illustrate the steps: 'Given a rectangular sheet of paper 9 inches by 12 inches, form a box by cutting congruent squares from the corners, folding up the sides and taping them to form an open box. To make a box with maximum capacity, how large should the squares be?' Here was my seven step path:

1. Identify the variable to be maximized or minimized (in this case, the volume V).
2. Write a formula for the variable (V=*length x breadth x height=lbh*).
3. Write the variable in terms of one variable only ($V= x(9-2x)(12-2x)$; here x is the side of one of the squares cut from each corner).
4. Differentiate the variable with respect to this variable. (In this case find $\frac{dv}{dx}$)
5. Set the derivative equal to zero and find the value of the independent variable at the turning point. ($\frac{dv}{dx} = 12x^2-84x+108$ which yields $x = 1.69, 5.30$)
6. Check by differentiating again and substituting these values of x in the second derivative, which value gives a maximum volume and which a minimum volume. (At $x = 1.69$, second derivative is $24*1.69-84< 0$, hence maximum.)
7. Go back to the question and give the specific information required. (In this case it was the size of the square cutouts which would be of side 1.69 cm and area approx. 2.85 square cm.)

Undoubtedly I was smoothing the path to good performance by providing the students with such a structured approach. As students used the seven steps for all the problems in the book, it appeared that most of them had mastered the content and that I had helped them to do so. There were some who never attained a degree of comfort with the topic – and whenever they approached me for a tutorial I would guide them through more problems with the same algorithm in my eagerness to prove that it was infallible. Very few of the students who had problems with this approach ever got comfortable with the topic, and they tended to shy away from this section in examinations.

Alan Wigley goes on to describe two different approaches to teaching and learning mathematics and how they seem to lie in watertight compartments. Here they are:

Exploration	Instruction
Invented methods	Given methods
Creative	Imitative
Reasoned	Rote
Informal	Formal
Progressive	Traditional
Open	Closed
Process	Content
Talking (pupil)	Talking (teacher)
Listening (teacher)	Listening (pupil)

The important point Wigley makes is about how lessons fall fully into one category or the other. For example, my lesson would clearly fall into the second category. Obviously, I needed to heed his advice to create classes that ensured conceptual understanding and enabled students to develop their own procedures. How could I do this in the time available? I paid heed to Wigley's advice to follow the 'challenging model' the features of which are given below:

The teacher presents a challenging context or problem and gives pupils time to work on it and make conjectures about methods or results. Often the teacher will have an aspect of the syllabus in mind, but this may not be declared to pupils at this stage.

An important word here is challenge*. The problem must be pitched at the right level, not too difficult, but more importantly, not too easy.*

A second important word is time*. It is crucial to give sufficient time for pupils to get into the problem – to recognise that it poses a challenge and that there may be a variety of approaches to it – so that discussion begins.*

Here again the role of the teacher is crucial – initially, in drawing out pupils' ideas. The syllabus may require the learning of more formal processes. The stimulus for this may be a harder mathematical problem and may require exposition by the teacher. However, the pupil will have the context of previous work to which more advanced techniques can be related.

A variety of techniques is used to help pupils to review their work, and to identify more clearly what they have learned and how it connects together. Longer term failure is dealt with by ensuring that any return to the same subject matter encourages a different point of view and does not just go over the same ground in the same way. The model places a strong emphasis on the learner gaining new insights, and the time required for reflection is considered to be fully justified.

The actual sub-unit began with group work on maxima and minima. The class was divided into groups of 4 and each group was given a problem. I deliberately used problems in which the dependent variable was a function of more than one variable.

In the initial 90 minute class, each group first worked on understanding the problem. After discussing what data the problem gave and what they were being asked to find out, the students decided on a method to represent the problem. At the end of the class, each group gave a short presentation on the problem: how they represented it, and how they used the model to collect data. For example, the group working on the box problem said that they would actually construct different boxes by cutting squares of different sizes from sheets of the given dimension. The group working on the swimming pool problem planned to create a simulation, and since exact rates could not be used, they would make a table of data using the given rates. For each group I reiterated the importance of explaining the need for optimization. Note that this class was spent in studying the problem and in listening to the other groups, and not on the solution to the problem.

Much before beginning the unit on maxima and minima, I had given a lot of emphasis to the concept of dependent and independent variable. I ensured that this idea was introduced while studying functions and revisited while creating tables and plotting graphs. In the next class, we used graphing software to understand the characteristics of turning points when the dependent variable was plotted against the independent variable. This software helped students to understand the reason why rate of change equals 0 at the turning point. They were also able to observe the sign change of the first derivative. Observations and conclusions were noted down in a worksheet which accompanied the exercise. In a class discussion following this exercise, the conclusions were discussed and noted down formally. This was followed by simple problems on maxima and minima from the textbook where the dependent variable was expressed in terms of one variable only.

Once the models were ready, the students were able to see the visual connect between the data given and the constraints specified. For example, in the box problem, students were able to measure and understand that the height of the box formed was the side of the square. This and other observations helped them write the volume in terms of the side *x* of the square cut out. I found this was crucial for them to understand that the given constraints allowed them to express the dependent variable in terms of one variable only. With the level of algebra that most students had drilled into them from high school, this was not a problem if one simply did a series of intricate steps that gave the desired result. But making the models helped students 'see' the implications of the constraints. Also, during the group discussion, peers observed and questioned and added their remarks. If a particular group was not able to proceed, suggestions were invited from other groups. Rarely did I have to intervene. Based on the common points from their problem, class mates were able to give constructive suggestions which helped the group in distress. Each group was able to arrive at the point where an expression for the dependent variable in terms of one single independent variable allowed a graph to be drawn. In the next class, we used these graphs and their learning so far on maxima and minima to complete the problem using differentiation and a formal algebraic procedure. For homework, each group had to do the remaining groups' problems.

Working in groups on a concrete or semi-concrete model helped students understand the problem and its solution. Certainly, some students still had doubts. But instead of countering their doubts with the same algorithm each time, I was able to use a variety of stimuli to understand as well as clarify their doubts. Eventually, I did share the 7 step plan with them. But this was after they had done a sufficient number of problems and they could connect each step to why they did this step. For the students who were comfortable with the topic, I encouraged to experiment with more difficult versions of the problem. For example, from an article on the box problem[ii] :-

a. If we use a square sheet of paper, does a common relationship exist between the side of this square paper and the side of the square cutout?
b. If the piece of paper we start with is an equilateral triangle, how do we cut the corners so that we can then fold up the sides and get a box with an equilateral triangle for base? What is the relationship between the side of the original equilateral triangle and the height of the lateral sides of the box in order for the box to have maximum volume?

Conclusion

While resources such as the Mathematics Teacher give teachers plenty of food for thought, it is the experience of modifying material to suit our particular need that makes the journey challenging and interesting. I was happy because I was able to incorporate elements of challenge, cooperative work and creativity, and at the same time preserve the rigor of the mathematics. I was also able to deliver a differentiated program of learning based on the student's mastery of the topic as well as comfort level with areas such as model making, simulating, use of graphing software, communicating and presenting. Finally, students were able to both reflect and critique on the experience. And where kindness dictated my 7 step approach, I was able to teach my students a better understanding of problem solving with this exercise. My thanks to Alan Wigley for having challenged me!

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ii 'Thinking Out of the Box' Mathematics Teacher Volume 95, No. 8, November 2002

SNEHA TITUS. a teacher of mathematics for the last twenty years has resigned from her full time teaching job in order to pursue her career goal of inculcating in students of all ages, a love of learning the logic and relevance of Mathematics. She works in the University Resource Centre of the Azim Premji Foundation. Sneha mentors mathematics teachers from rural and city schools and conducts workshops using the medium of small teaching modules incorporating current technology, relevant resources from the media as well as games, puzzles and stories which will equip and motivate both teachers and students. She may be contacted on sneha.titus@azimpremjifoundation.org